V. A. Smirnov, I. M. Smirnova

REMARKABLE PLANE CURVES

(Properties. Construction. Modeling)

2026

PREFACE

The book discusses remarkable curves in the plane, their properties, construction, and modeling.

It is intended for first-year university students studying advanced geometry. The corresponding course was developed and implemented into the educational process at the Institute of Mathematics and Informatics of Moscow Pedagogical State University. It was based on an elective course developed within the framework of the "Mathematics. Engineers of the Future" project by the Prosveshcheniye Publishing House.

In this course, students will be able to:

- become familiar with classical curves and their properties;
- illustrate these curves and model them using computer software;
- use various methods for analytically defining curves.

This course in analytical geometry will also:

- increase students' motivation to study mathematics;
- accommodate different levels of preparation and diverse interests of first-year students;
- fill gaps in students' knowledge of geometry;
- prepare them for advanced mathematics courses.

It consists of three modules.

The first module is devoted to curves as sets of points satisfying a given condition. It examines one of the fundamental ways of defining curves on a plane, which consists of specifying a property that the points of these curves satisfy.

Figures composed of all points, and only those points, that satisfy a given property are given the special name “geometric loci.”

In the school geometry curriculum, the perpendicular bisector of a segment and the angle bisector are considered as examples of geometric loci.

In this course, the concept of a geometric locus is used to define various curves on a plane, including the parabola, ellipse, hyperbola, named curves, and others.

The possibilities of using the computer program GeoGebra to model curves as geometric loci are demonstrated.

The second module examines curves as trajectories of moving points. Such curves include the cycloid, epicycloids and hypocycloids, cardioid, astroid, Steiner curve, and others.

The possibilities of using the computer program GeoGebra to model curves as trajectories of moving points are demonstrated.

The third module considers the analytical representation of curves in the plane, including the representation of curves by:

- an equation in Cartesian coordinates;
- parametric equations;
- equations in polar coordinates.

In the fourth module, we present curves on the sphere and Poincaré models of the Lobachevsky plane.

In addition to theoretical material, each module of this course contains exercises in computation, proof, construction, and modeling.

At the end of the book, answers and solutions to the exercises are provided.

MODULE 1. CURVES AS SET OF POINTS

1. Parabola

Fix a line *d* in the plane and a point *F* not lying on it. A ***parabola*** is the locus of points in the plane that are equidistant from a given line *d* and a given point *F*. The line *d* is called the ***directrix***, and the point *F* is called the ***focus*** of the parabola (Fig. 1.1). The point *G* on a parabola nearest to its directrix is called its vertex.

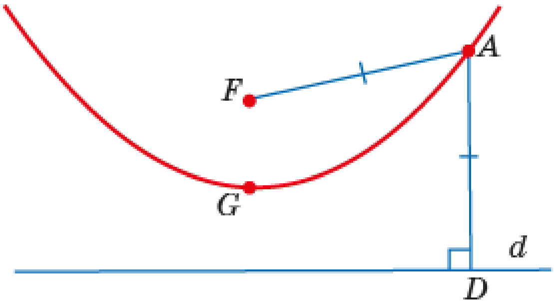


Fig. 1.1

The name «parabola» was introduced by Apollonius of Perga an ancient Greek mathematician, one of the three great geometers of antiquity (alongside Euclid and Archimedes) who lived in the 3rd century BC. It comes from the Greek word παραβολή (parabolē), which means «application», «comparison», or «approximation».

To construct a parabola, one can use the GeoGebra computer program, whose workspace is shown in Figure 1.2.

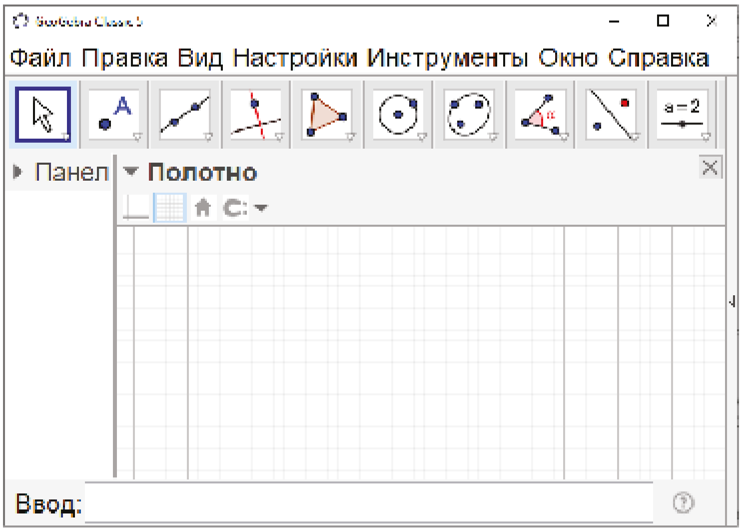


Fig. 1.2

Using the "Line" tool, draw a line. Label it *d*.

Using the "Point" tool, mark a point not lying on this line. Label it *F*.

Using the "Slider" tool, create a slider *r* that varies from 0 to 5.

Using the "Circle with Center and Radius" tool, construct a circle *b* with center *F* and radius *r*.

Draw a line *c* at a distance *r* from line *d*.

Using the "Intersect" tool, find the intersection point *A* of the constructed circle *b* and line *c*. This point *A* will be equidistant from point *F* and line *d*.

In the properties of the obtained point, select the "Trace On" option.

When the slider value is changed, the obtained point will leave a trace in the shape of a parabola (Fig. 1.3).

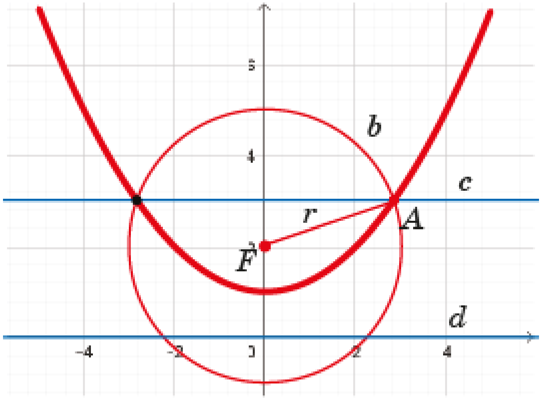


Fig. 1.3

Let’s demonstrate another way to construct a parabola in the computer program GeoGebra.

Using the "Line" tool, draw a line. Label it *d*.

Using the "Point" tool, mark a point not lying on this line. Label it *F*.

Mark a point *D* on line *d*.

Using the "Perpendicular Line" tool, draw a line *c* through point *D*, perpendicular to line *d*.

Using the "Perpendicular Bisector" tool, draw the perpendicular bisector *b* of segment *DF*.

Using the "Intersect" tool, find the intersection point *A* of the constructed lines *b* и *c*. This point will be equidistant from point *F* and line *d*.

As point *D* moves along line *d*, this point *A* will leave a trace in the shape of a parabola (Fig. 1.4).

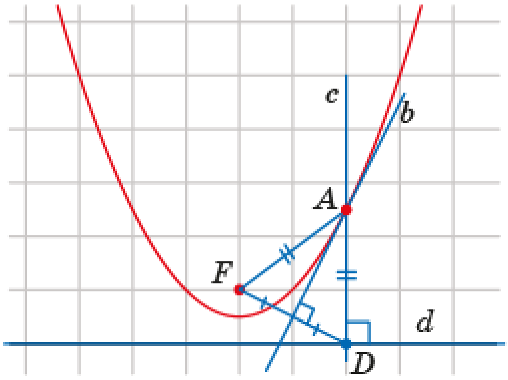


Fig. 1.4

In this program, a parabola can also be obtained using the "Parabola" tool. To do this, select the "Parabola" tool. Indicate point *F* and line *d*. A parabola with focus *F* and directrix *d* will appear on the screen (Fig. 1.5).

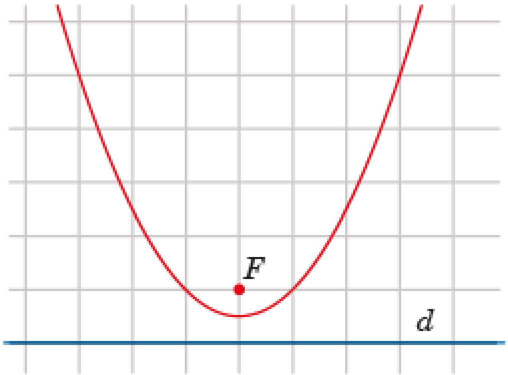


Fig. 1.5

Mark some point A on the parabola. Using the "Distance or Length" tool, measure the distances from this point to the focus F and to the directrix d. By moving point A along the parabola, one can verify that these distances are equal.

The parabola divides the plane into two regions. The focus of the parabola lies in one of them, and the directrix lies in the other. We will call the first region the interior, and the second the exterior.

Let us prove that for any point B located in the exterior region of the parabola, its distance to the focus is greater than its distance to the directrix (Fig. 1.6).

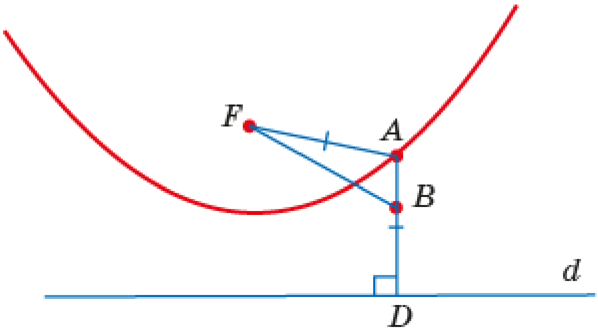


Fig. 1.6

Connect points B and F with a segment. Through point B draw a line perpendicular to line d. Let A, D denote its intersection points with the parabola and the directrix. Using the triangle inequality, we obtain $BF > AF - AB = AD - AB = BD$. Finally, we get $BF > BD$.

By analogy with the definition of a tangent to a circle, we define the concept of a tangent to a parabola.

A ***tangent*** to a parabola is a line that has exactly one common point with the parabola, while all other points lie in the exterior region (Fig. 1.7).

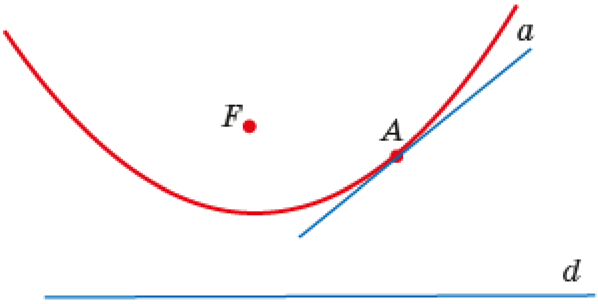


Fig. 1.7

Theorem. Let point *A* belong to a parabola with focus *F* and directrix *d*. Drop a perpendicular *AD* from this point to the directrix. Then the line passing through point *A* and containing the angle bisector of angle *FAD* is tangent to the parabola (Fig. 1.8).

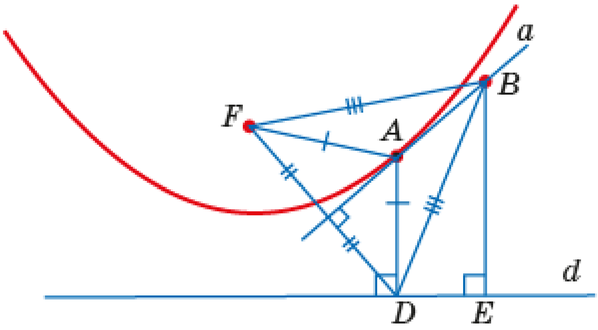
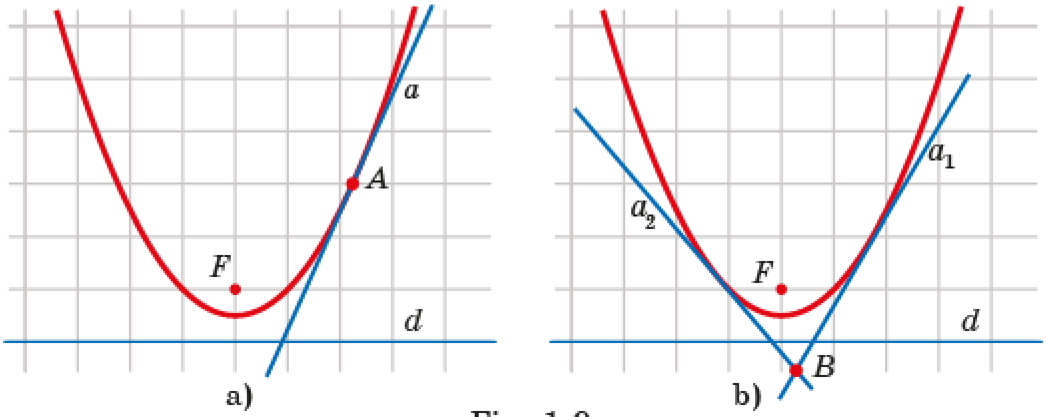


Fig. 1.8

Proof. Triangle *ADF* is isosceles (*AD* = *AF*). Recall that in an isosceles triangle, the angle bisector drawn to the base is also the median and the altitude. Consequently, the line *a* containing the bisector of angle *FAD* is the perpendicular bisector of segment *DF*. For an arbitrary point *B* on line *a*, distinct from *A*, drop a perpendicular *BE* to line *d*. Then the equality *BF* = *BD* holds. Segment *BD* is oblique to line *d*, while segment *BE* is perpendicular. Hence, the inequality *BD* > *BE* holds, from which it follows that *BF* > *BE*. Therefore, point *B* does not belong to the parabola and lies in the exterior region.

The tangent to a parabola passing through a given point can be constructed in GeoGebra. To do this, use the "Tangents" tool, specifying the point and the parabola (Fig. 1.9).



a) b)

Fig. 1.9

One of the remarkable properties of the parabola is its optical property, which is used in the construction of telescope surfaces, parabolic antennas, etc.

Optical property. If a light source is placed at the focus of a parabola, the rays, after reflecting off the parabola, travel in a single direction perpendicular to the directrix.

Proof. We use the fact that light reflects off a curve in the same way as off the tangent line drawn at the point of incidence. Suppose a ray of light emanates from the focus *F* and reflects off the parabola at point *A* (Fig. 1.10).

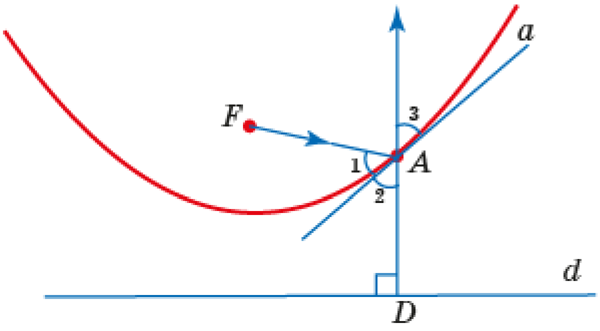


Fig. 1.10

Denote by 1 and 2 the angles into which the tangent divides angle *FAD*, and by 3 the angle vertical to angle 2. Then angles 1 and 3 are equal. Since the angle of incidence equals the angle of reflection, the reflected ray will travel in the direction of ray *DA*, i.e., in the direction perpendicular to the directrix.

Exercises

1. Draw a parabola with a given focus and directrix (Fig. 1.11).

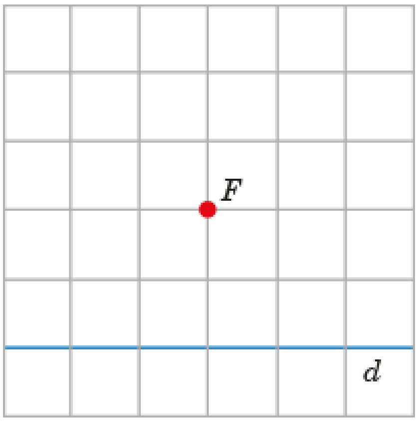


Fig. 1.11

2. Construct a parabola using the GeoGebra computer program.
3. Prove that for points *C* located in the interior region of a parabola, the distance to the focus is less than the distance to the directrix.
4. For a given focus *F* and directrix *d* of a parabola construct the tangent to the parabola passing through a given point (Fig. 1.12).

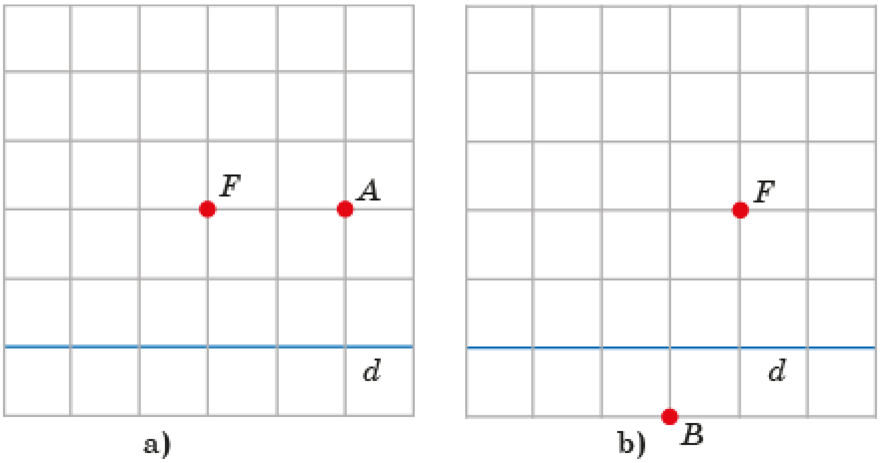


Fig. 1.12

5. How many tangents to a parabola can be drawn through a point: a) lying on the parabola; b) located in the exterior region; c) located in the interior region?

6. For two given points on a parabola and its directrix find the position of the focus of the parabola (Fig. 1.13). Determine how many solutions the problem has.

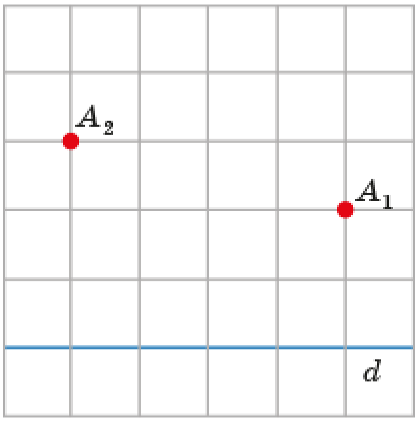


Fig. 1.13

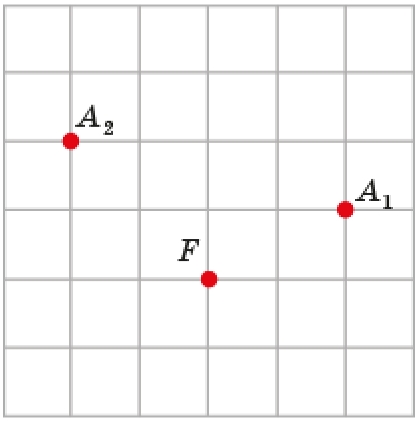


Fig. 1.14

7. For two given points on a parabola and its focus construct the directrix of the parabola (Fig. 1.15). Determine how many solutions the problem has.

8. Find the locus of centers of circles that pass through a given point A and are tangent to a given line b (Fig. 1.16).

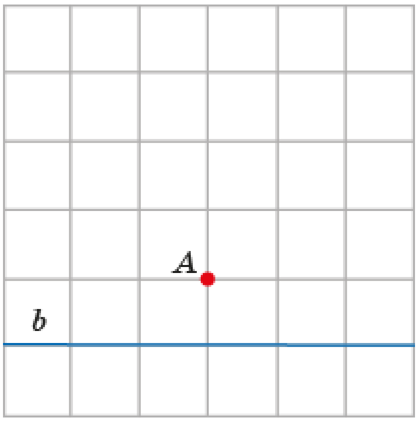


Fig. 1.15

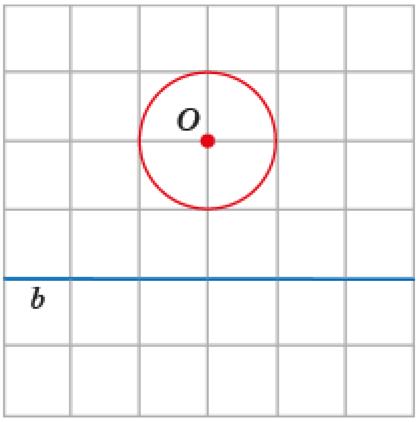


Fig. 1.16

9. Find the set of vertices C of triangles ABC with two equal sides ($AC = BC$), where vertex A is fixed and vertex B lies on a given line b not passing through point A (Fig. 1.15).

10. Find the lokus of points B for which the difference between the distance to a given point A and the distance to a given line b is equal to 1 (Fig. 1.15).

11. Find the locus of points equidistant from a given line and a given circle (Fig. 1.16).

12. Find the locus of centers of circles that are tangent to a given line and a given circle: a) externally; b) internally (Fig. 1.16).

13. Find the angle between two tangents to a parabola drawn through a point on the directrix. Prove it.

14. For two given intersecting lines a and b, and a point C not lying on them construct a circle tangent to these lines and passing through the given point (Fig. 1.17).

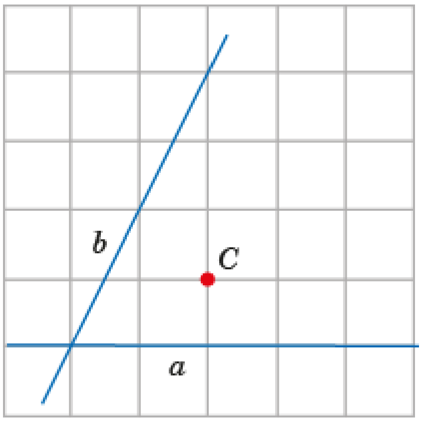


Fig. 1.17

15. Given a parabola with focus F and directrix d. Tangents a_1 and a_2 are drawn from point A. Prove that the angles under which the segments AA_1 and AA_2 of the tangents are viewed from the focus F of the parabola are equal (Fig. 1.18).

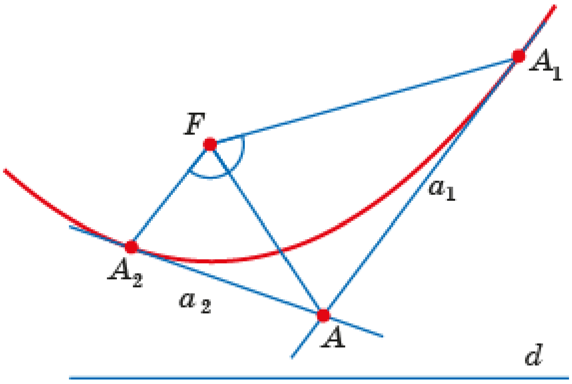


Fig. 1.18

16. Given a parabola with focus F and directrix d. Tangents a_1 and a_2 are drawn from point A. Through point B on the parabola, located inside the angle a_1Aa_2, tangent b is drawn. B_1 and B_2 are its intersection points with tangents a_1 and a_2, respectively (Fig. 1.19).

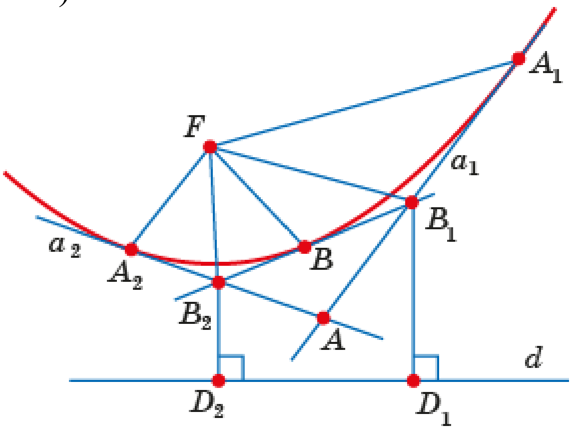


Fig. 1.19

Prove that the following are independent of the position of point B:

a) the value of angle B_1FB_2, find it, if $\angle A_1AA_2 = \alpha$;

b) the length of the orthogonal projection D_1D_2 of segment B_1B_2 onto the directrix d;

c) the sum of the lengths of segments AB_1 and AB_2.

2. Ellipse

Let us fix points F_1 and F_2 on a plane. An ***ellipse*** is the set of all points on the plane for which the sum of the distances to the given points F_1 and F_2 is equal to a given number c, which is greater than the distance between these points. The points F_1 and F_2 themselves are called the ***foci*** of the ellipse (Fig. 2.1).

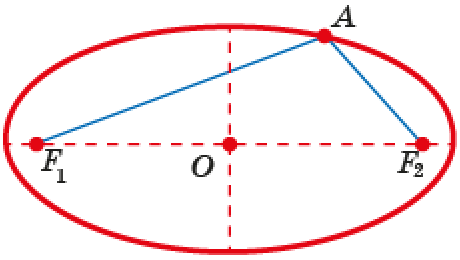


Fig. 2.1

Thus, for points A of the ellipse, the following equality holds:

$$AF_1 + AF_2 = c, \text{ where } c > F_1F_2.$$

The segment whose endpoints lie on the ellipse and which contains its foci is called the ***major axis*** of the ellipse.

The midpoint of the major axis of an ellipse is called its ***center***.

The segment perpendicular to the major axis, passing through the center, and whose endpoints lie on the ellipse, is called the ***minor axis*** of the ellipse.

The name «ellipse» was introduced by Apollonius of Perga. It comes from the ancient Greek word ἔλλειψις (elleipsis), which means «deficiency».

Johannes Kepler (1571–1630) discovered that the planets of the Solar System move not in circles, as previously thought, but in ellipses, with the Sun located at one of the foci of these ellipses.

To construct an ellipse, one can use the computer program GeoGebra.

For example, set $c = 7$ (Fig. 2.2).

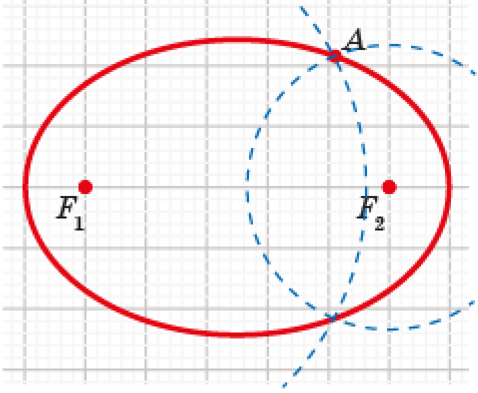


Fig. 2.2

Using the "Point" tool, mark points F_1, F_2 such that the distance between them is 5.

Using the "Slider" tool, create a slider a that varies from 0 to 6.

Using the "Circle with Center and Radius" tool, construct a circle with center F_1 and radius a, and a circle with center F_2 and radius $c - a$.

Using the "Intersect" tool, find the intersection points of the constructed circles.

In the properties of the obtained points, select the option "Trace On".

By changing the slider value, the resulting points will leave a trace in the form of an ellipse.

Let’s demonstrate another way to construct an ellipse in the computer program GeoGebra.

Set $c = 7$. Using the "Point" tool, mark points F_1, F_2 such that the distance between them is 5.

Using the "Circle with center and radius" tool, construct a circle with center F_1 and radius c. Label it d.

Mark a point D on line d.

Draw a segment F_1D. Label it b.

Using the "Perpendicular Bisector" tool, draw the perpendicular bisector a of segment F_2D.

Using the "Intersect" tool, find the intersection point A of the constructed lines b и a.

For this point A the following equality will hold: $AF_1 + AF_2 = c$.

As point D moves along circle d, this point A will leave a trace in the shape of an ellipse (Fig. 2.3).

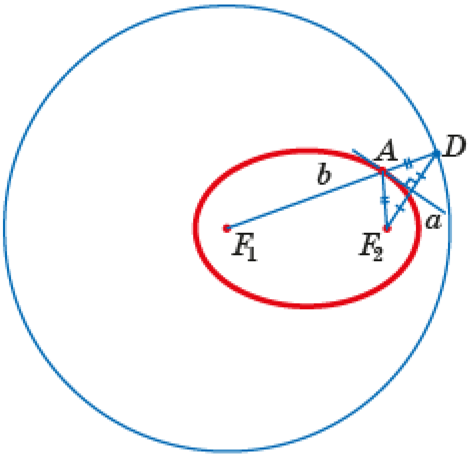


Fig. 2.3

In this program, an ellipse can also be constructed using the "Ellipse" tool.

To do this, select the "Ellipse" tool. Indicate points F_1, F_2, and some point A.

An ellipse with foci F_1, F_2 passing through point A will appear on the screen (Fig. 2.4).

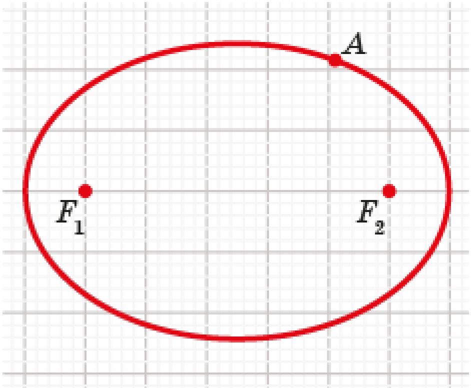


Fig. 2.4

Mark some point A on the ellipse. Using the "Distance or Length" tool, measure the distances from this point to the foci F_1, F_2. By moving point A along the ellipse, one can verify that the sum of these distances remains constant.

The ellipse with foci F_1, F_2 and constant c divides the plane into two regions. One of these regions contains the foci of the ellipse. We will call this region the interior, and the other the exterior.

Let us prove that for any point B located in the exterior region of the ellipse, the sum of the distances to the foci is greater than c.

Connect point B with points F_1, F_2 by line segments. Denote by A the point of intersection of segment BF_2 with the ellipse (Fig. 2.5).

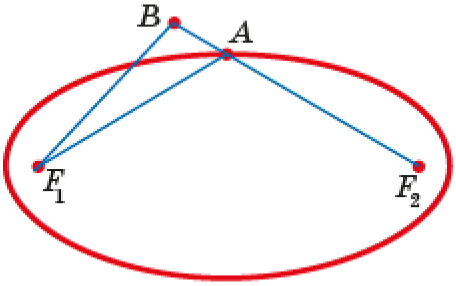


Fig. 2.5

Then, $BF_1 + BF_2 = BF_1 + BA + AF_2 > AF_1 + AF_2 = c$.

By analogy with the definition of a tangent to a circle, we define the concept of a tangent to an ellipse.

A **tangent** to an ellipse is a line that has exactly one common point with the ellipse (Fig. 2.6).

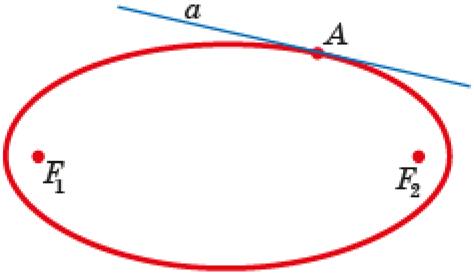


Fig. 2.6

Theorem. Let point A belong to the ellipse with foci F_1, F_2. Then the line passing through point A and containing the bisector of the angle supplementary to angle F_1AF_2 is tangent to the ellipse.

Proof. On the extension of segment F_1A, lay off segment AF' equal to AF_2 (Fig. 2.7).

Triangle AF_2F' is isosceles ($AF_2 = AF'$). Consequently, the line a containing the bisector of angle F_2AF' is the perpendicular bisector of segment F_2F'. For any point B on line a other than A, the equality $BF_2 = BF'$ holds. Using the triangle inequality, we obtain:

$$BF_1 + BF_2 = BF_1 + BF' > F_1F' = F_1A + AF_2 = c.$$

Hence, point B does not belong to the ellipse. Therefore, line a has exactly one common point with the ellipse, i.e., it is a tangent.

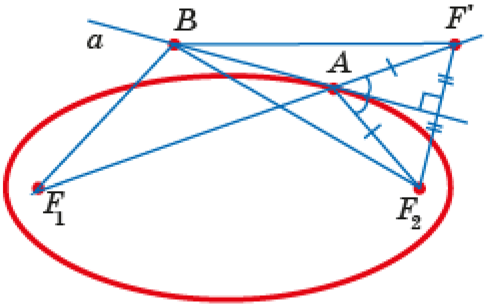
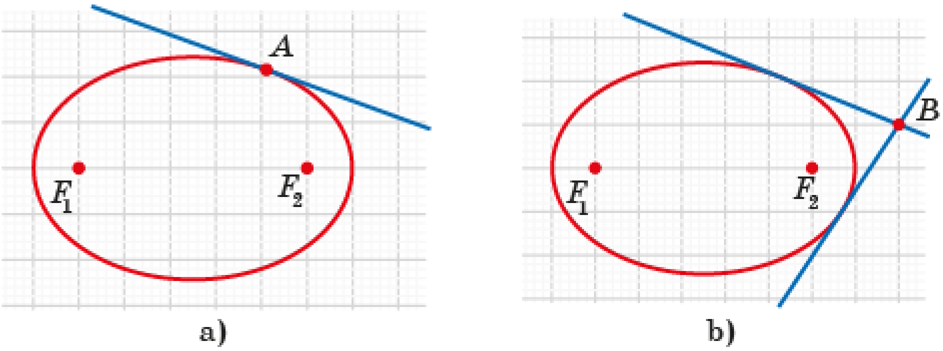


Fig. 2.7

A tangent to an ellipse passing through a given point can be constructed in GeoGebra. For this, use the "Tangents" tool, specifying the point and the ellipse (Fig. 2.8).



a) b)

Fig. 2.8

One of the remarkable properties of the ellipse is the following optical property.

Optical property. If a light source is placed at one focus of the ellipse, then the rays reflected off the ellipse will pass through the other focus (Fig. 2.9).

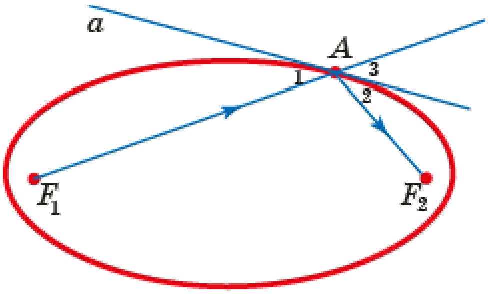


Fig. 2.9

Proof. Let angles 1 and 2 be formed by segments F_1A, F_2A and the tangent a passing through point A (Fig. 2.9). Let angle 3 be vertical to angle 1. Since the tangent contains the bisector of the angle supplementary to angle F_1AF_2, angles 2 and 3 are equal. Consequently, angles 1 and 2 are equal. Suppose a ray of light emanates from focus F_1 and reflects off the ellipse at point A. Since the angle of incidence equals the angle of reflection, the reflected ray will travel along line AF_2 and pass through focus F_2.

Exercises

1. Draw an ellipse with given foci and constant $c = 6$ (Fig. 2.10).

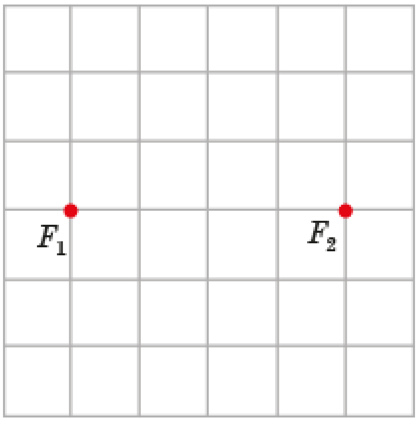


Fig. 2.10

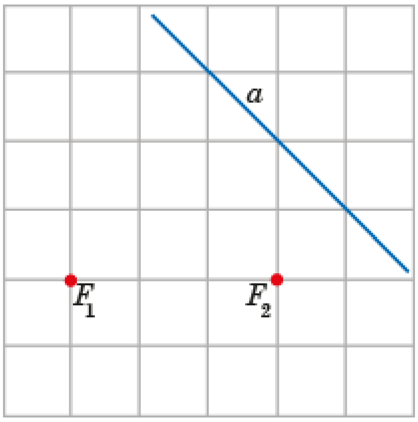


Fig. 2.11

2. Construct an ellipse using the computer program GeoGebra.
3. Prove that for points A located in the interior region of the ellipse, the sum of the distances to the foci is less than the constant c of this ellipse.
4. For given foci F_1, F_2 of an ellipse and a tangent a find the point of tangency (Fig. 2.11).
5. For given foci F_1, F_2 of an ellipse draw the tangent to the ellipse passing through a given point A on the ellipse (Fig. 2.12).

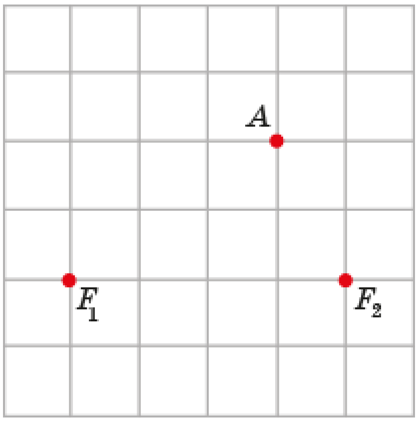


Fig. 2.12

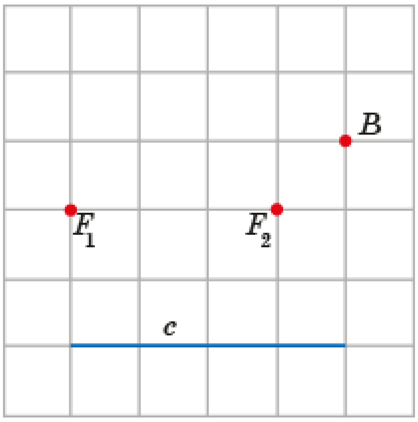


Fig. 2.13

6. For given foci F_1, F_2 and constant c of an ellipse draw the tangent to the ellipse passing through a given point B located in the exterior region (Fig. 2.13).
7. How many tangents to an ellipse can be drawn through a point: a) belonging to the ellipse; b) located in the exterior region; c) located in the interior region?
8. For two given points A_1, A_2 on the ellipse, constant c, and one focus F_1 find the position of the second focus (Fig. 2.14).

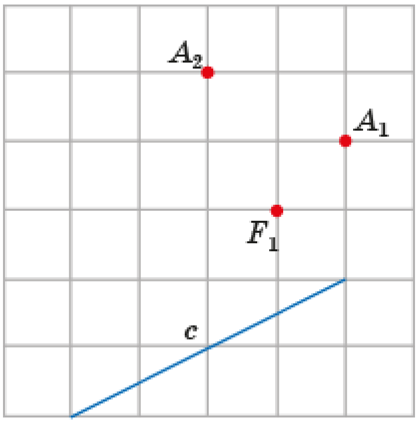


Fig. 2.14

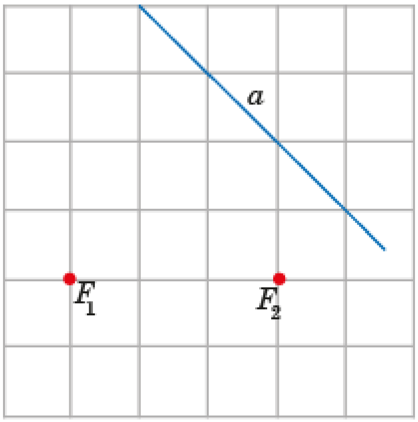


Fig. 2.15

9. For given foci F_1, F_2 of an ellipse and a tangent a find the constant c of this ellipse (Fig. 2.15).

10. For a given focus F_1 of an ellipse, constant c, tangent a, and point of tangency A find the position of the second focus (Fig. 2.16).

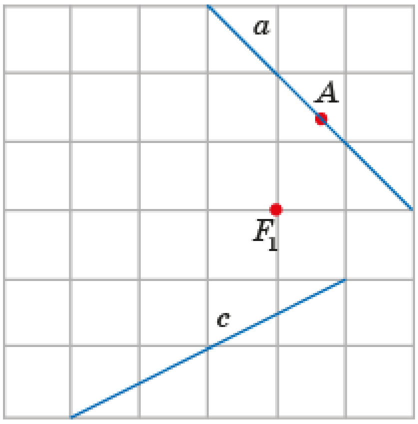


Fig. 2.16

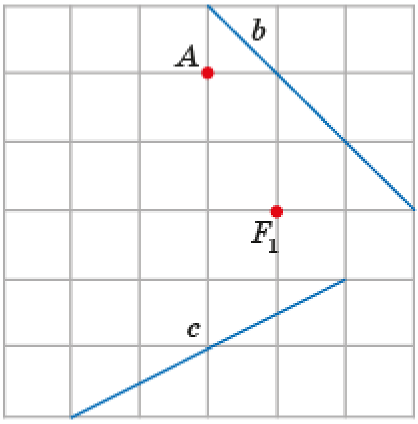


Fig. 2.17

11. For a given focus F_1 of an ellipse, constant c, tangent b, and point A on the ellipse not lying on this tangent find the position of the second focus (Fig. 2.17).

12. For given points A and B, describe the locus of vertices C of triangles ABC whose perimeter is 4 (Fig. 2.18).

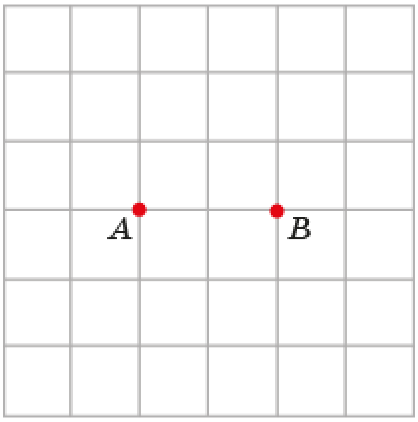


Fig. 2.18

13. Describe the locus of points equidistant from a given circle and a given point P located inside this circle and distinct from its center (Fig. 2.19).

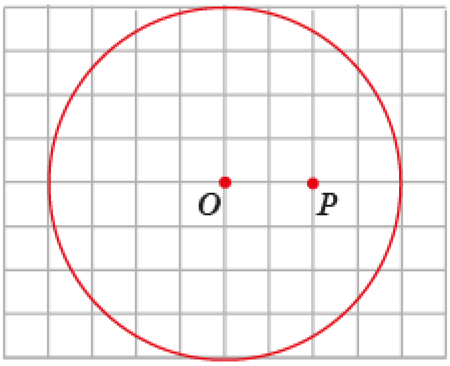


Fig. 2.19

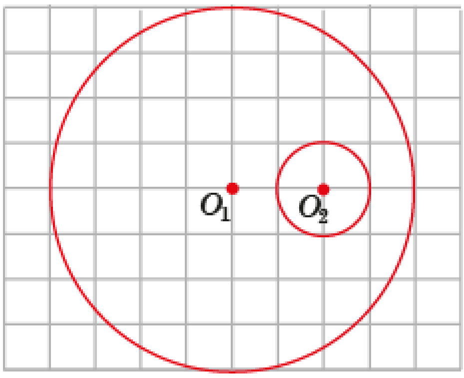


Fig. 2.20

14. Describe the locus of centers of circles that are tangent to a given circle and pass through a point P located inside this circle and distinct from its center (Fig. 2.19).

15. Describe the locus of points equidistant from two given circles, one of which lies inside the other (Fig. 2.20).

16. One circle lies inside another circle. Describe the locus of centers of circles that are externally tangent to the inner circle and internally tangent to the outer circle (Fig. 2.20).

17. Describe the locus of centers of circles that are internally tangent to two circles, one of which lies inside the other (Fig. 2.20).

18. Given an ellipse with foci F_1, F_2. Tangents a_1 and a_2 are drawn from point A. Prove that the angles under which the segments AA_1 and AA_2 of the tangents are viewed from the focus of the ellipse are equal (рис. 2.21).

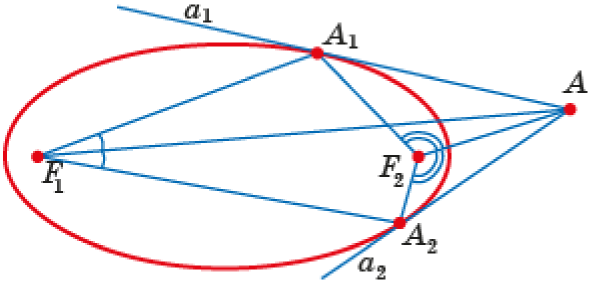


Fig. 2.21

19. Given an ellipse with foci F_1, F_2. Tangents a_1 and a_2 are drawn from point A. Let A_1, A_2 be the corresponding points of tangency. Prove that the angles F_1AA_1 and F_2AA_2 are equal (рис. 2.22).

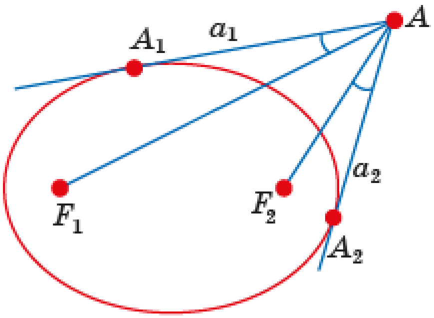


Fig. 2.22

20. Given an ellipse with foci F_1, F_2 and the constant c, $F_1F_2 = d$. Prove that the locus of points A from which the ellipse is seen at a right angle (the angle between the tangents a_1, a_2 drawn from point A) is a circle with center O of the ellipse O (рис. 2.23). Find the radius R of this circle.

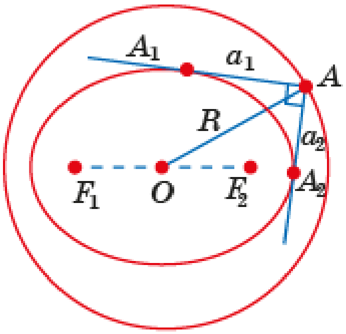


Fig. 2.23

21. Given an ellipse with foci F_1, F_2. Tangents a_1 and a_2 are drawn from point A. Through point B on the ellipse, located inside the angle a_1Aa_2, tangent b is drawn. B_1 and B_2 are its intersection points with tangents a_1 and a_2, respectively (Fig. 2.24).

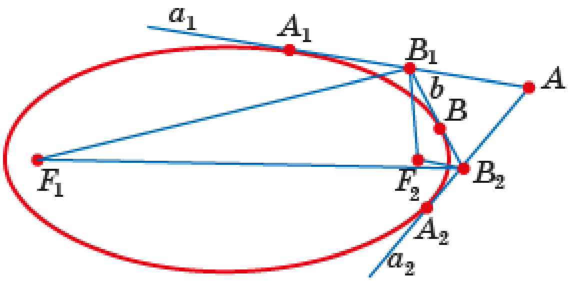


Fig. 2.24

Prove that the values of the angles $B_1F_1B_2$, $B_1F_2B_2$ under which the segment B_1B_2 of the tangent b is viewed from the foci of the ellipse are independent of the position of point B and $\angle B_1F_1B_2 + \angle A_1AA_2 < 180°$, $\angle B_1F_2B_2 + \angle A_1AA_2 < 180°$.

3. Hyperbola

Let us fix points F_1, F_2 on the plane. A ***hyperbola*** is the set of all points on the plane for which the absolute difference of the distances to the given points F_1, F_2 is equal to a given positive number c, which is less than the distance between these points. The F_1, F_2 themselves are called the ***foci*** of the hyperbola (Fig. 3.1).

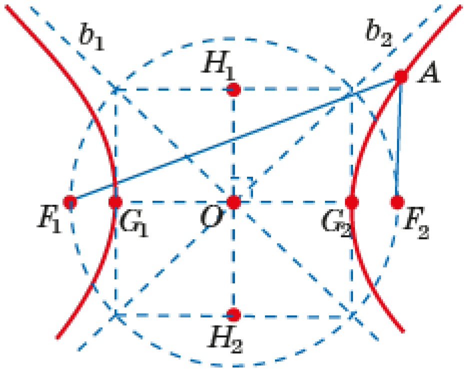


Fig. 3.1

A hyperbola consists of two separate curves, which are called branches. For their points, one of the following equalities holds:

$$AF_1 - AF_2 = c, AF_2 - AF_1 = c.$$

The points G_1, G_2 on the two branches of a hyperbola that are closest to each other are called the ***vertices***.

The segment with endpoints at the vertices of a hyperbola is called the ***major axis***.

The midpoint O of the major axis is called the ***center*** of the hyperbola.

The segment H_1H_2 is called the ***minor axis***.

The lines b_1,b_2 are called the asymptotes of a hyperbola.

The name «hyperbola» was introduced by Apollonius of Perga. It comes from the ancient Greek ὑπερβολή (hyperbolē), which means «excess».

To obtain a hyperbola, you can use the GeoGebra computer program.

For example, put $c = 2$.

Using the "Point" tool, mark points F_1, F_2 such that the distance between them is 4.

Using the "Slider" tool, create a slider a that varies from 3 to 6.

Using the "Circle with Center and Radius" tool, construct a circle with center F_1 and radius a, and a circle with center F_2 and radius $a - c$.

Using the "Intersect" tool, find the intersection points of the constructed circles.

In the properties of the obtained points, select the "Trace On" option.

When changing the slider value, the obtained points will leave a trace in the form of a branch of the hyperbola (Fig. 3.2).

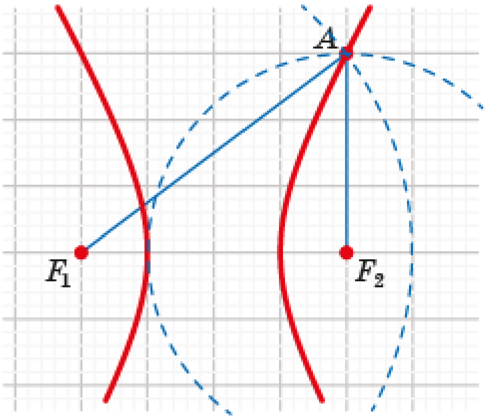


Fig. 3.2

The other branch of the hyperbola is obtained in a similar manner.

Let’s demonstrate another way to construct a hyperbola in the computer program GeoGebra.

Set $c = 2$. Using the "Point" tool, mark points F_1, F_2 such that the distance between them is 4.

Using the "Circle with center and radius" tool, construct a circle with center F_1 and radius c. Label it d.

Mark a point D on line d.

Draw a ray F_1D. Label it b.

Using the "Perpendicular Bisector" tool, draw the perpendicular bisector a of segment F_2D.

Using the "Intersect" tool, find the intersection point A of the constructed lines b и a.

For this point A the following equality will hold: $AF_1 - AF_2 = c$.

As point D moves along circle d, this point A will leave a trace in the shape of a branch of the hyperbola (Fig. 3.3).

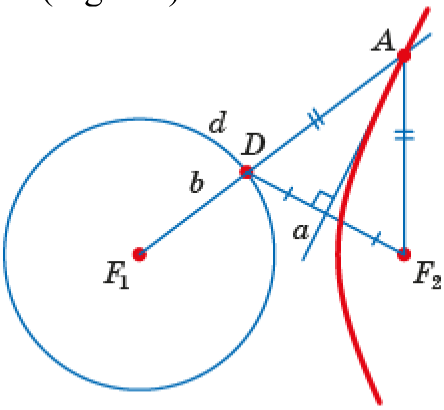


Fig. 3.3

In this program, a hyperbola can also be obtained using the "Hyperbola" tool.

To do this, select the "Hyperbola" tool. Indicate points F_1, F_2 and some point A.

A hyperbola with foci F_1, F_2 passing through point A will appear on the screen (Fig. 3.4).

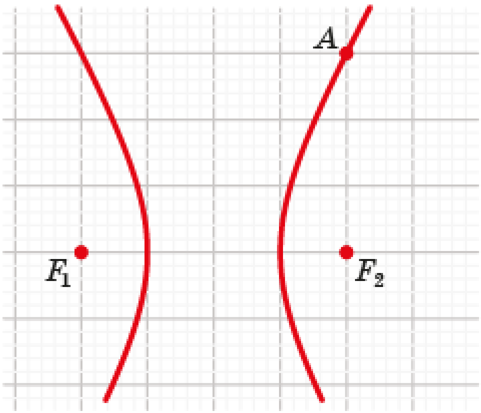


Fig. 3.4

Mark some point A on the hyperbola. Using the "Distance or Length" tool, measure the distances from this point to the foci F_1, F_2. By moving point A along the hyperbola, one can verify that the difference of these distances does not change.

Each branch of the hyperbola with foci F_1, F_2 and constant c divides the plane into two regions. One of these regions contains the focus of the hyperbola. We will call this region the interior, and the other the exterior relative to this branch of the hyperbola.

In Figure 3.5, the interior part corresponding to focus F_2 of the hyperbola is shaded.

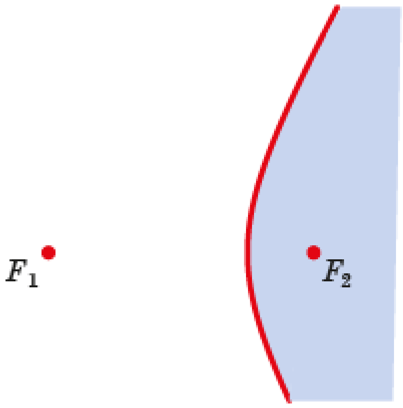


Fig. 3.5

By analogy with the definition of a tangent to a parabola, we define the concept of a tangent to a hyperbola.

A **tangent** to a hyperbola with foci F_1, F_2 and constant c is a line that has exactly one common point with the hyperbola, while all its other points lie in the exterior region.

Theorem. Let point A belong to the hyperbola with foci F_1, F_2. Then the line passing through point A and containing the bisector of angle F_1AF_2 is tangent to this hyperbola.

Proof. Consider a point A on the branch of the hyperbola corresponding to focus F_2. On segment F_1A, mark segment AF' equal to AF_2 (Fig. 3.6).

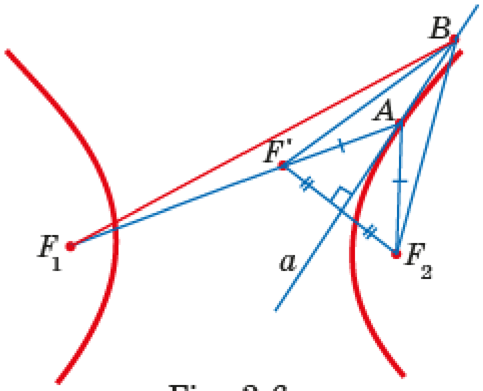


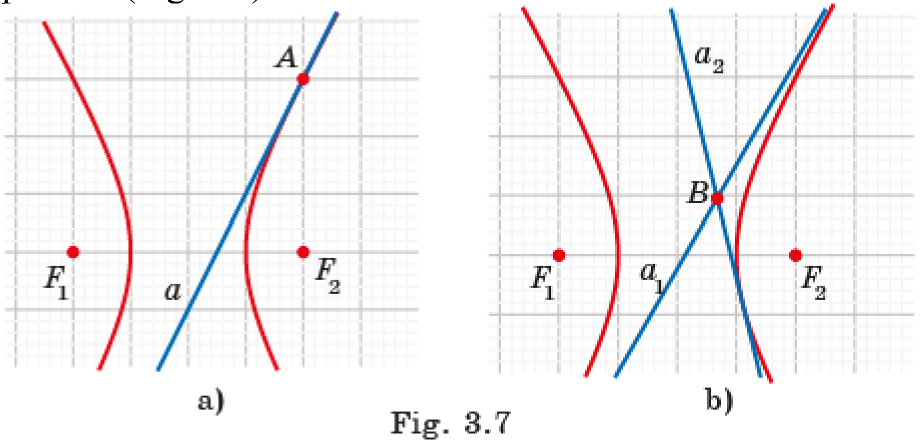
Fig. 3.6

Triangle AF_2F' is isosceles ($AF_2 = AF'$). Consequently, the line a containing the bisector of angle F_2AF' is the perpendicular bisector of segment F_2F'. For an arbitrary point B on line a, different from A, the equality $BF_2 = BF'$ holds. We use the triangle inequality. We obtain

$$BF_1 - BF_2 = BF_1 - BF' < F_1F' = F_1A - AF_2 = c.$$

Thus, point B does not belong to the hyperbola. Therefore, line a has exactly one common point with the hyperbola, i. e., it is a tangent.

A tangent to a hyperbola passing through a given point can be obtained in the GeoGebra program. To do this, use the "Tangents" tool, specifying the point and the hyperbola (Fig. 3.7).



a) b)

Fig. 3.7

One of the remarkable properties of the hyperbola is the following optical property.

Optical property. If a light source is placed at one focus of the hyperbola, the rays, after reflecting off the hyperbola, will appear to emanate from the other focus (Fig. 3.8).

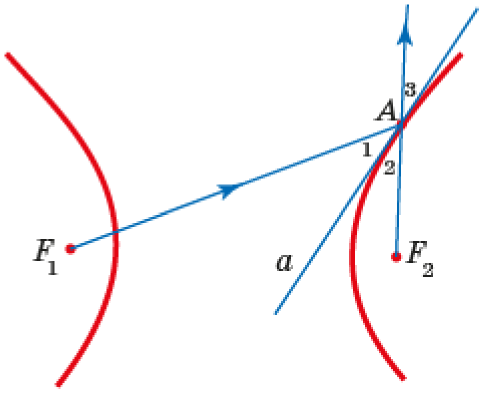


Fig. 3.8

Proof. Denote by 1 and 2 the angles formed by the segments F_1A, F_2A and the tangent a passing through point A of the hyperbola. Denote by 3 the angle vertical to angle 2. Since the tangent contains the bisector of angle F_1AF_2, angles 1 and 2 are equal. Consequently, angles 1 and 3 are equal. Suppose a light ray emerges from focus F_1 and reflects off the hyperbola at point A. Since the angle of incidence equals the angle of reflection, the reflected ray will travel along line F_2A as if it originated from focus F_2.

Exercises

1. Draw a hyperbola with given foci and constant c (Fig. 3.9).

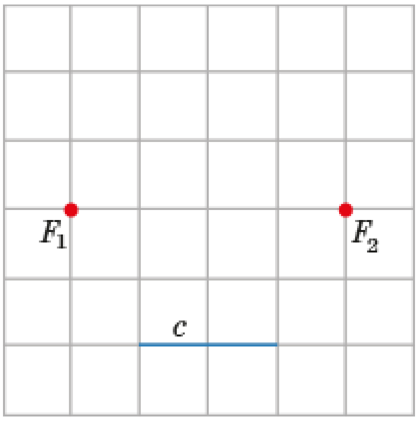


Fig. 3.9

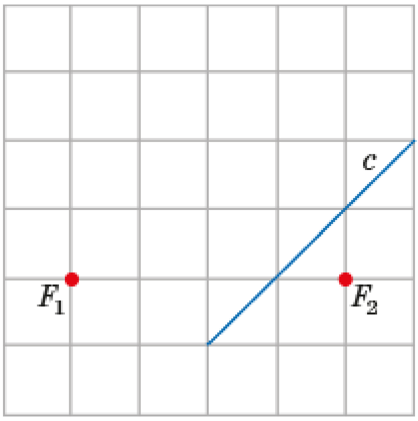


Fig. 3.10

2. Obtain a hyperbola using the GeoGebra computer program.
3. Prove that for points B located in the exterior region of a hyperbola branch with foci F_1, F_2, the inequality $|BF_1 - BF_2| < c$ holds.
4. For given foci F_1, F_2 of a hyperbola and a tangent line a find the constant c and the point of tangency (Fig. 3.10).
5. For given foci F_1, F_2 of a hyperbola draw the tangent to the hyperbola passing through a given point on the hyperbola (Fig. 3.11).

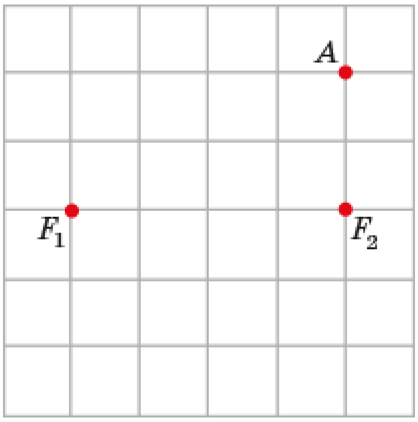


Fig. 3.11

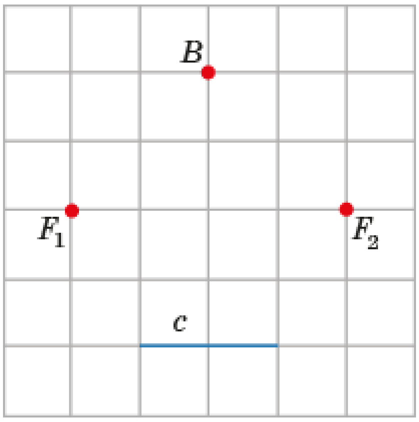


Fig. 3.12

6. For given foci F_1, F_2 and constant c of a hyperbola draw the tangent to the hyperbola passing through a given point B located in the exterior region (Fig. 3.12).

7. How many tangents to a hyperbola can be drawn through a point: a) belonging to the hyperbola; b) located in the exterior region; c) located in the interior region?

8. For two given points A_1, A_2 on the hyperbola, the constant c, and one focus F_1 find the position of the second focus (Fig. 3.13).

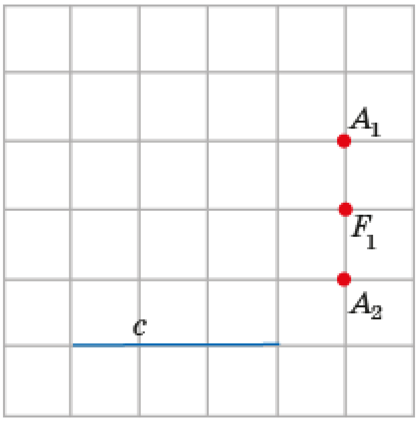


Fig. 3.13

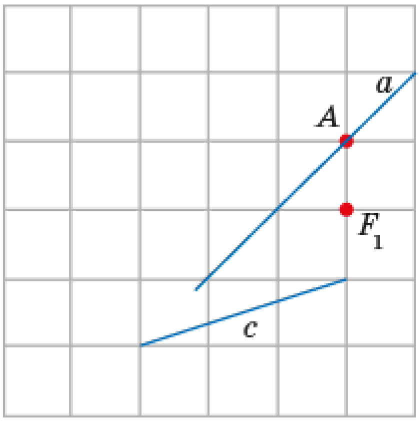


Fig. 3.14

9. For a given focus F_1 of a hyperbola, the constant c, a tangent line a, and the point of tangency A find the position of the second focus (Fig. 3.14).

10. For a given focus F_1 of a hyperbola, the constant c, a tangent line b, and a point A on the hyperbola not lying on this tangent find the position of the second focus (Fig. 3.15).

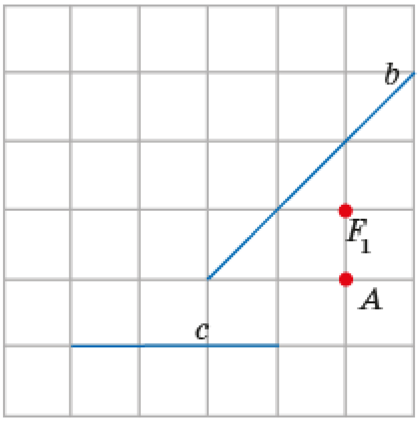


Fig. 3.15

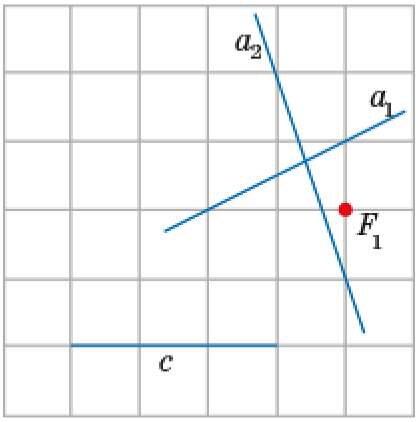


Fig. 3.16

11. For a given focus F_1 of a hyperbola, the constant c, and two tangent lines a_1, a_2 (Fig. 3.16) construct the second focus of this hyperbola.

12. Describe the locus of points equidistant from a given circle and a given point P located outside this circle (Fig. 3.17).

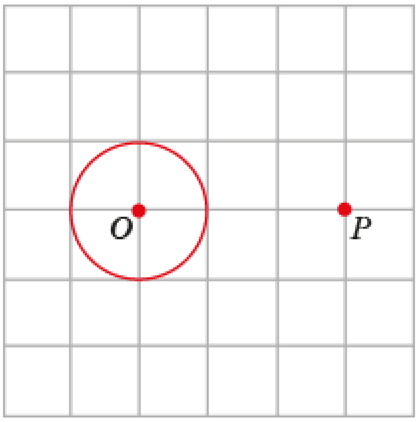


Fig. 3.17

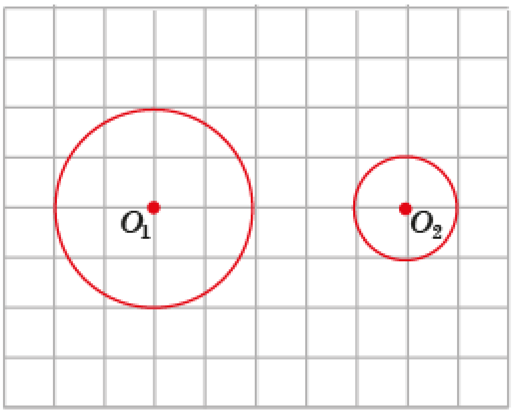


Fig. 3.18

13. Describe the locus of centers of circles that are externally tangent to a given circle and pass through a point P located outside this circle (Fig. 3.17).

14. Describe the locus of centers of circles that are internally tangent to a given circle and pass through a point P located outside this circle (Fig. 3.17).

15. Describe the locus of centers of circles that are externally tangent to two given circles (Fig. 3.18).

16. Describe the locus of centers of circles that are internally tangent to two given circles (Fig. 3.18).

17. Given a hyperbola with foci F_1, F_2. Tangents a_1 and a_2 are drawn from point A. Prove that the angles under which the segments AA_1 and AA_2 of the tangents are viewed from the focus of the hyperbola are equal (Fig. 3.19).

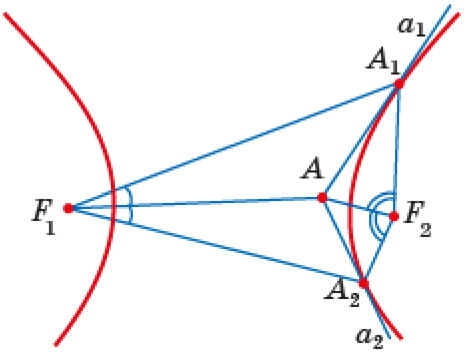


Fig. 3.19

18. Given a hyperbola with foci F_1, F_2. Tangents a_1 and a_2 are drawn from point A. Let A_1, A_2 be the corresponding points of tangency. Prove that the angles F_1AA_1 and F_2AA_2 are equal (рис. 3.20).

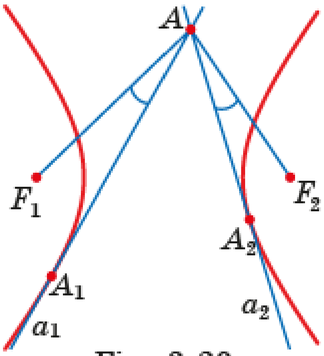


Fig. 3.20

19. Given a hyperbola with foci F_1, F_2 and the constant c, $F_1F_2 = d$. Prove that the locus of points A, for which the tangents to the hyperbola are perpendicular, is a circle with center O, excluding the points belonging to the asymptotes of this hyperbola (рис. 3.21). Find the radius r of this circle.

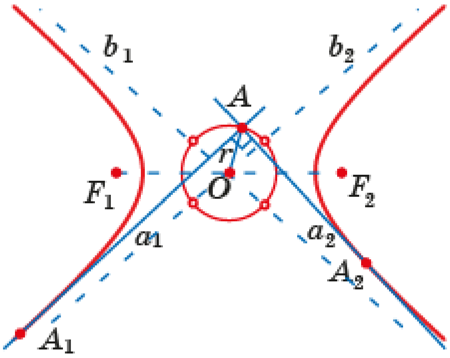


Fig. 3.21

20. Given a hyperbola with foci F_1, F_2. Tangents a_1 and a_2 are drawn from point A. Through point B on the hyperbola, located inside the angle a_1Aa_2, tangent b is drawn. B_1 and B_2 are its intersection points with tangents a_1 and a_2, respectively (Fig. 3.22). Prove that the values of the angles $B_1F_1B_2$, $B_1F_2B_2$ under which the segment B_1B_2 of the tangent b is viewed from the foci of the hyperbola are independent of the position of point B and $\angle B_1F_2B_2 + \angle A_1AA_2 > 180°$.

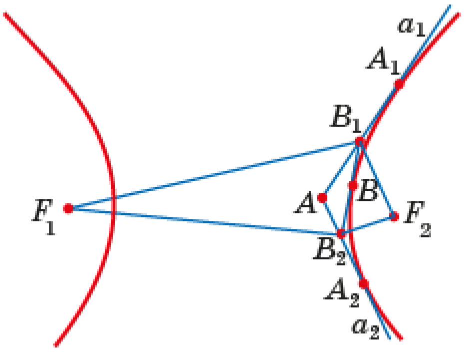


Fig. 3.20

4. Famous named curves

Here we will consider several examples of famous named curves obtained as geometric loci.

1. Conchoid of Nicomedes. Draw a line c and a point P at a distance d from this line. Choose any point C on line c. Draw line PC. On this line, mark points A, B such that $AC = BC = l$.

The conchoid of Nicomedes is the locus of points A and B corresponding to all possible points C on line c (Fig. 4.1).

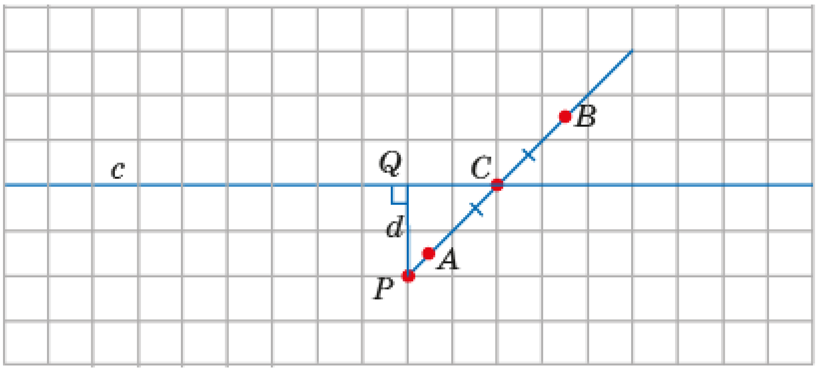


Fig. 4.1

If $l < d$, the conchoid of Nicomedes has the shape shown in Figure 4.2.

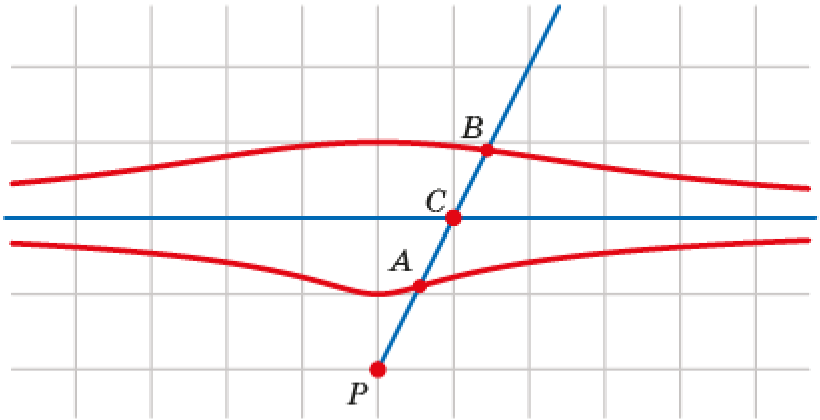


Fig. 4.2

If $l = d$, the conchoid of Nicomedes has the shape shown in Figure 4.3.

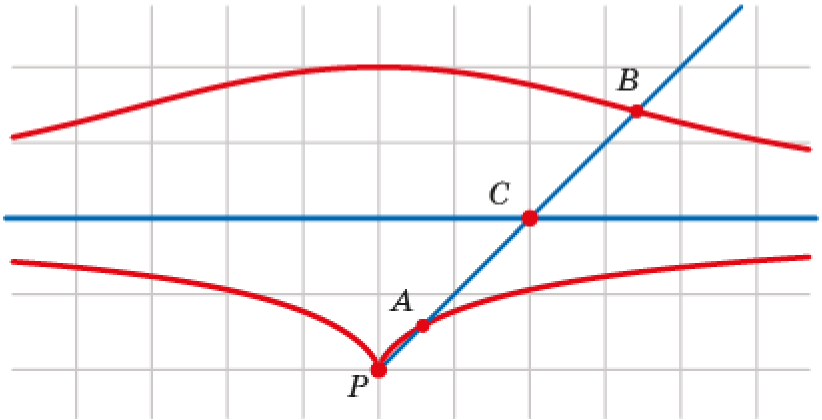


Fig. 4.3

If $l > d$, the conchoid of Nicomedes has the shape shown in Figure 4.4.

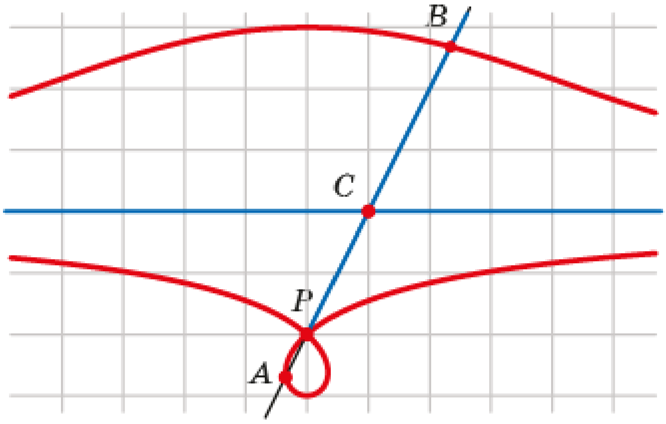


Fig. 4.4

2. Pascal's snail. Draw a circle c with center O and radius R. Fix a point P on it. Choose any point C on this circle. Draw line PC. On this line, mark points A, B such that $AC = BC = l$. The Pascal's snail is the locus of points A and B corresponding to all possible points C on circle c (Fig. 4.5).

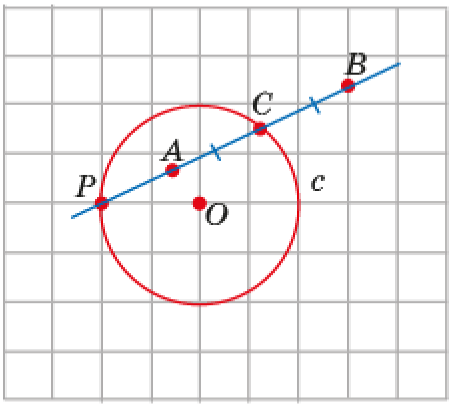


Fig. 4.5

If $l < d$, the Pascal's snail has the shape shown in Figure 4.6.

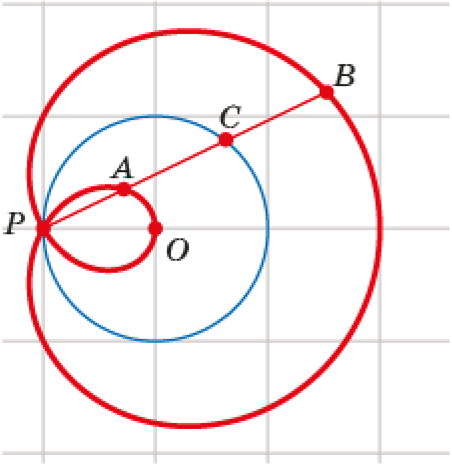


Fig. 4.6

If $l = d$, the Pascal's snail has the shape shown in Figure 4.7.

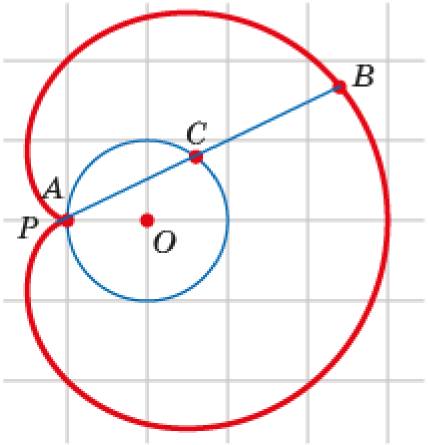


Fig. 4.7

If $l > d$, the Pascal's snail has the shape shown in Figure 4.8.

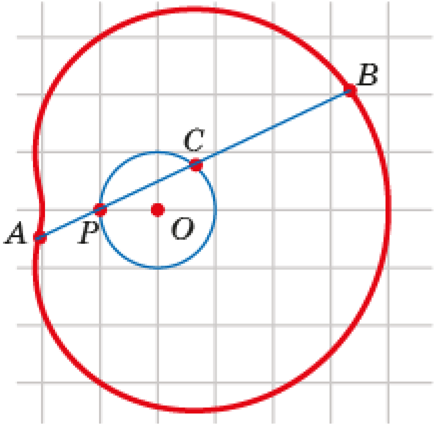


Fig. 4.8

3. Strophoid of Torricelli. Draw a line c and a point P at a distance d from this line. From point P, draw a perpendicular PQ to line c. Choose any point C on line c. Draw line PC. On this line, mark segments $AC = BC = CQ$.

The strophoid is the locus of points A and B corresponding to all possible points C on line c (Fig. 4.9).

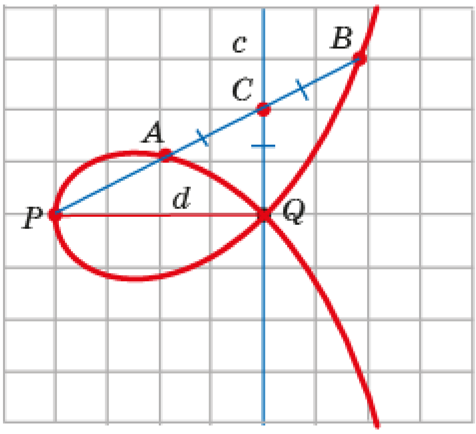


Fig. 4.9

4. Cissoid of Diocles. Draw a circle with center *O*. Fix a point *P* on it. Draw the diameter *PQ* of this circle. Through point *Q*, draw a tangent line *c*. From point *P*, draw a perpendicular *PQ* to line *c*. Choose any point *C* on line *c*. Draw line *PC*. Denote by *B* its intersection point with the circle. On this line, mark point *A* such that *PA* = *BC*.

The cissoid of Diocles is the locus of points *A* and *B* corresponding to all possible points *C* on line *c* (Fig. 4.10).

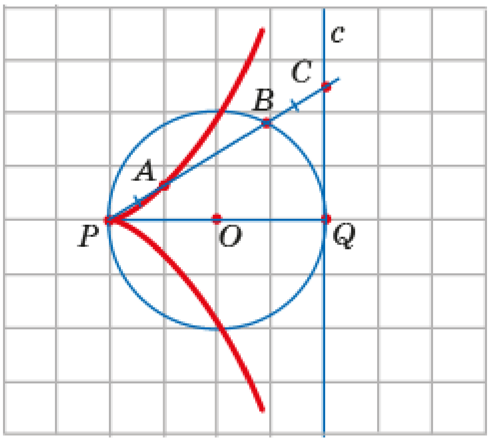


Fig. 4.10

5. Kappa of Gutschoven. Draw a line *c*. Fix a point *O* on it. Choose any point *C* on line *c*. Construct a circle with center at point *C* and radius *r*. Through point *O*, draw tangents to the constructed circle. Denote the points of tangency as A_1 and A_2.

The kappa is the geometric locus of points A_1 and A_2 corresponding to all possible points *C* on line *c* (Fig. 4.11).

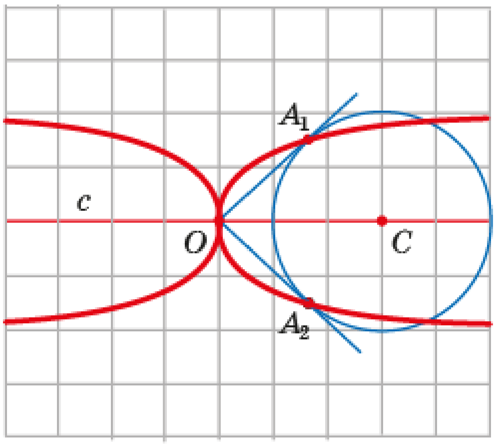


Fig. 4.11

6. Lemniscate of Bernoulli. Fix points F_1 and F_2. The lemniscate of Bernoulli is the geometric locus of points *A* in the plane for which the product of the distances AF_1 and AF_2 equals the square of half the distance between points F_1 and F_2. The points F_1 and F_2 themselves are called ***foci*** (Fig. 4.12). Denote by *a* half the distance between points F_1 and F_2. Then points *A* of the lemniscate of Bernoulli satisfy the equation $AF_1 \cdot AF_2 = a^2$.

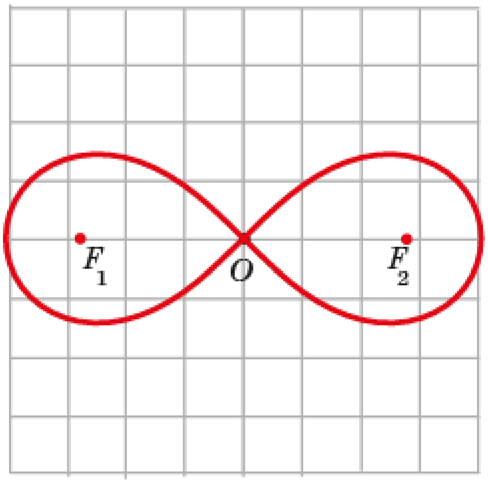


Fig. 4.12

7. Circle of Apollonius. Fix points A and B. The circle of Apollonius is the geometric locus of points C for which the ratio of the distances CA and CB equals a given positive number $c \neq 1$. Figure 4.13 shows the circle of Apollonius for which $c = 2$.

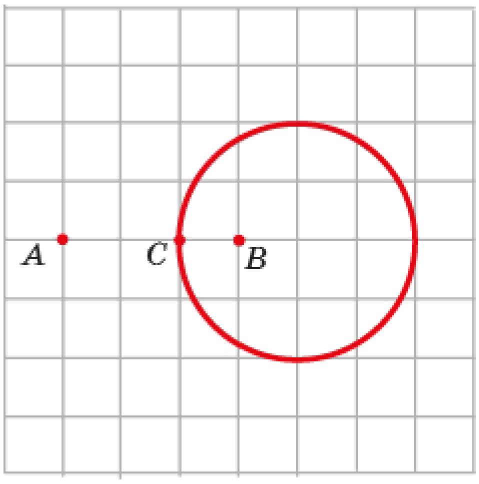


Fig. 4.13

8. Maclaurin Trisectrix. Fix points A and B. Draw line AB. Rotate line AB around point A by an angle t counterclockwise. Denote the resulting line as a. Rotate line AB around point B by an angle $3t$ counterclockwise. Denote the resulting line as b (Fig. 4.14).

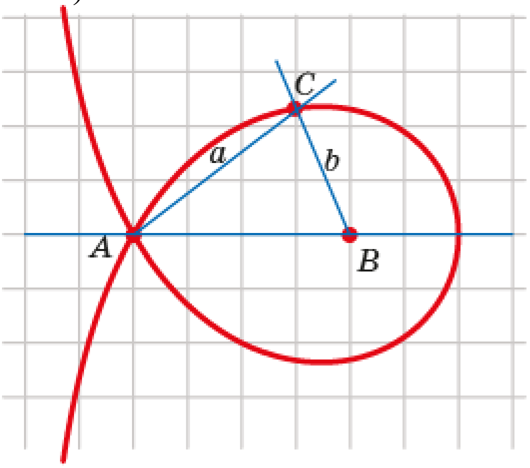


Fig. 4.14

The Maclaurin trisectrix is the geometric locus of points C at the intersection of lines a and b, corresponding to all possible values of the angle t

9. Versiera of Agnesi. Draw a circle with center O and diameter PQ. Through points P and Q, draw tangents p and q respectively. Mark a point A on diameter PQ. Through it, draw a line a perpendicular to this diameter. Denote by B_1, B_2 the intersection points of this line with the circle. Draw lines PB_1, PB_2. Denote by C_1, C_2 their intersection points with the tangent passing through point Q. Through points C_1, C_2, draw lines perpendicular to the tangents. Denote by A_1, A_2 their intersection points with line a. The versiera of Agnesi is the geometric locus of points A_1, A_2 corresponding to all possible points A on diameter PQ (Fig. 4.15).

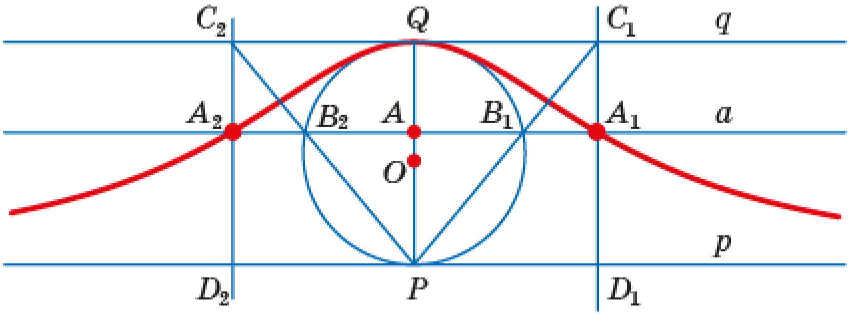


Fig. 4.15

The ***pedal curve*** of a given curve is the locus of the feet of the perpendiculars drawn from a fixed point to the tangents of that curve.

10. Let’s construct the pedal curve of the given circle with respect to the given point A, located outside the circle (Fig. 4.16).

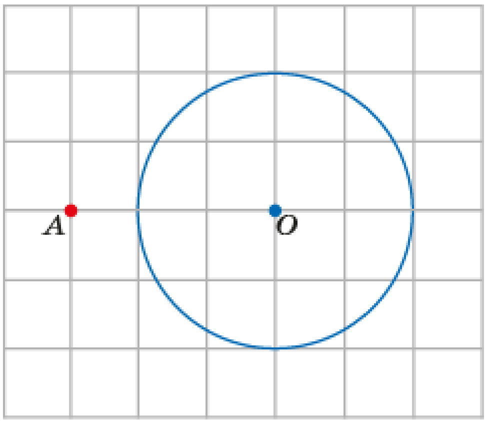


Fig. 4.16

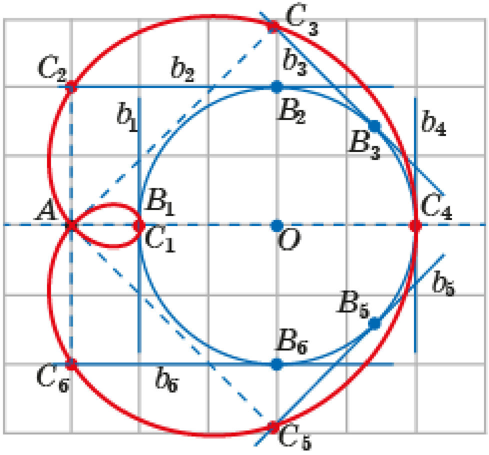


Fig. 4.17

Figure 4.17 shows the tangents b_1, ..., b_6, the points of tangency B_1, ..., B_6, the feet of the perpendiculars C_1, ..., C_6 dropped onto these tangents, and the curve passing through these feet.

The construction of this curve can be carried out in the computer program GeoGebra. Specifically, using the "Circle with center and radius" tool, we construct a circle with center O and radius 2. Using the "Point" tool, we mark point A at a distance of 3 from center O. We mark point B on the circle. Using the "Tangent" tool, we draw a tangent to the circle through point B. Using the "Perpendicular line" tool, we draw a line through point A perpendicular to this tangent. Using the "Intersect" tool, we construct point C, the intersection of the tangent line and the line perpendicular to it. In the properties of point C, we select the "Trace On" option. By moving point B along the circle, point C moves and leaves a trace in the form of the desired pedal curve (Fig. 4.18).

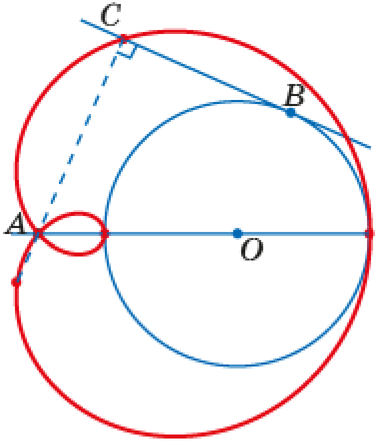


Fig. 4.18

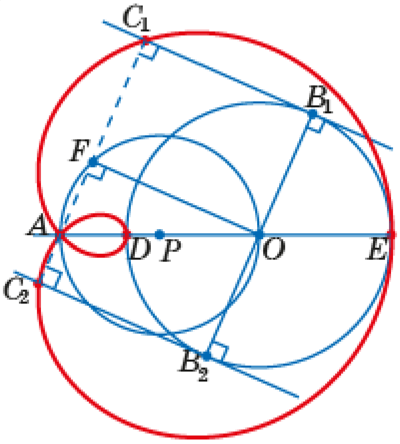


Fig. 4.19

Prove that this curve is a Pascal snail. Let's consider a circle with diameter $DE = 2l$ and center O (Fig. 4.19). Draw tangents to it at diametrically opposite points B_1, B_2. From point A, drop perpendiculars $AC_1, AC_2$to these tangents. Let Dbe the intersection point of line $C_1C_2$with the circle having diameter AO. The quadrilaterals $OB_1C_1D$and OB_2C_2D are rectangles. Consequently, $DC_1 = DC_2 = l$. Hence, points C_1, C_2lie on the limaçon of Pascal.

11. Construct the pedal curve of the given parabola with respect to the given point A located on the directrix (Fig. 4.20).

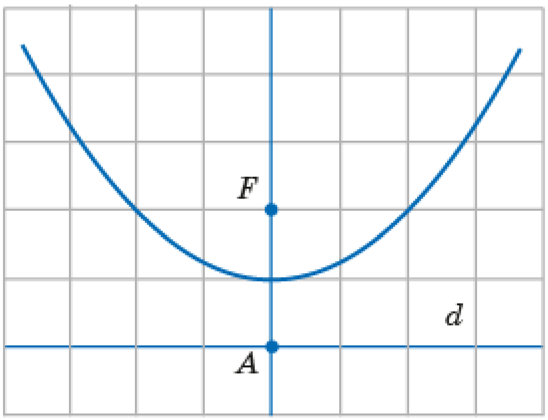


Fig. 4.20

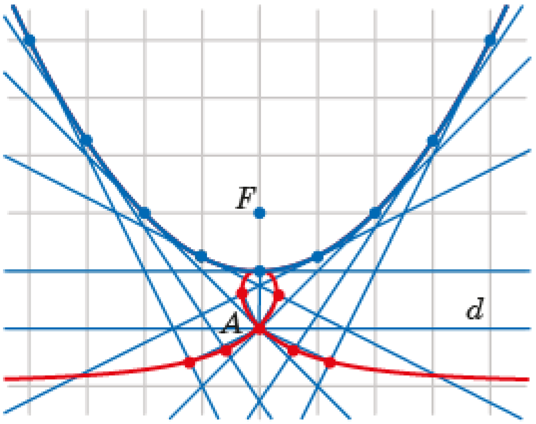


Fig. 4.21

Figure 4.21 shows the tangents b_1, ..., b_6, the points of tangency B_1, ..., B_6, the feet of the perpendiculars C_1, ..., C_6 dropped onto these tangents, and the curve passing through these feet.

The construction of this curve can be carried out in the computer program GeoGebra. Draw a line d and mark a point F at a distance of 2 from this line. Using the "Perpendicular line" tool, draw a line through point F (the focus of the parabola) perpendicular to the directrix. Using the "Intersect" tool, find their intersection point A. Using the "Parabola" tool, construct a parabola with focus F and directrix d. Mark a point B on the parabola. Using the "Tangent" tool, draw a tangent to the parabola through point B. Using the "Perpendicular line" tool, draw a line through point A perpendicular to this tangent. Using the "Intersect" tool, construct point C, the intersection of the tangent and the line perpendicular to it. In the properties of point C, select the "Trace On" option. By moving point B along the parabola, point C moves and leaves a trace in the form of the desired curve (Fig. 4.22).

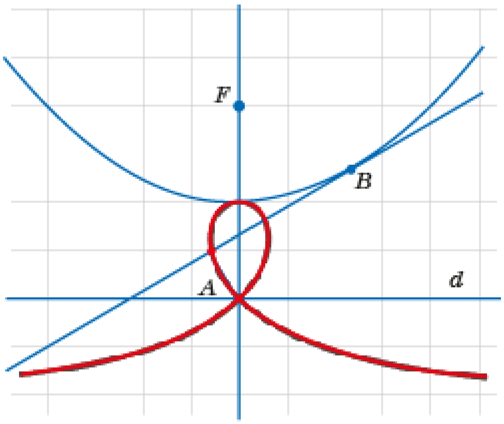


Fig. 4.22

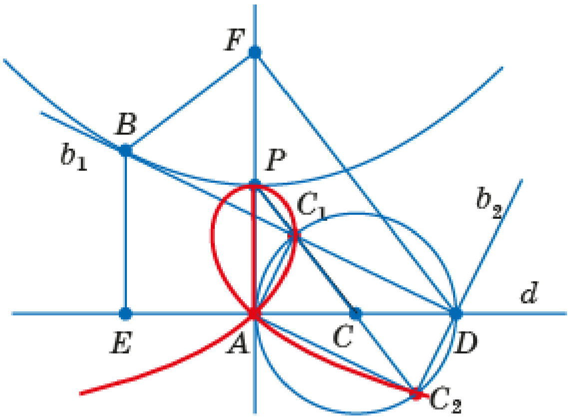


Fig. 4.23

Prove that this curve is a strophoid. Let points C_1, C_2 belong to the strophoid with vertex at the vertex P of the parabola (Fig. 4.23). Then PC is the midsegment of triangle AFD. Hence, it is parallel to FD, and $\angle ADF = \angle ACP$. Since angle ADC_1 is equal to half of angle ACP, the line DC_1 contains the angle bisector of $\angle ADF$. Consequently, it is tangent to the parabola, and point C_1 belongs to the pedal curve of the parabola. The line DC_2 contains the bisector of the angle adjacent to $\angle ADF$. Hence, it is also a tangent, and point C_2 belongs to the pedal curve of the parabola.

Exercises

1. Obtain the conchoid of Nicomedes using GeoGebra. Use: a) $d = 2$, $l = 1$; b) $d = l = 2$; c) $d = 2$, $l = 3$.
2. Obtain the Pascal’s snail using GeoGebra. Use: a) $R = 1$, $l = 1$; b) $R = 1$, $l = 2$; c) $R = 1$, $l = 3$.
3. Obtain the strophoid using GeoGebra.
4. Obtain the cissoid of Diocles using GeoGebra.
5. Obtain the kappa curve using GeoGebra.
6. Obtain the lemniscate of Bernoulli using GeoGebra. Use $a = 2$.
7. Obtain the circle of Apollonius using GeoGebra.
8. Obtain the Maclaurin trisectrix using GeoGebra.
9. Obtain the versiera of Agnesi using GeoGebra.
10. Consider a circle with center O and diameter AB (Fig. 4.24). Point P lies on the diameter AB and is distinct from the center O. Draw a chord AC through point A. Draw a line c through point C that is perpendicular to the line AB. Draw a line d through point P that is parallel to the line AC. Let D be the point of intersection of lines c and d. The locus of all such points D, corresponding to all possible positions of point C, is called ***Cramer's curve***. Illustrate this curve on graph paper or in the computer program GeoGebra.

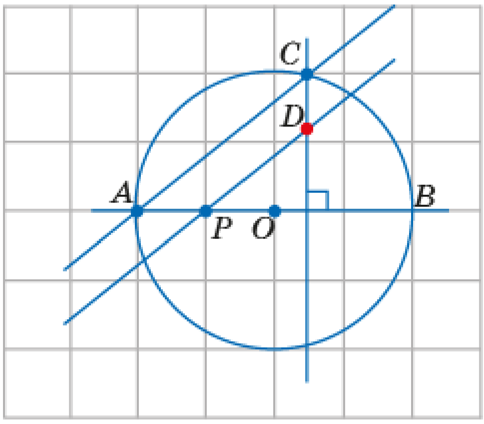


Fig. 4.24

11. Prove that the Maclaurin trisectrix is a special case of the Cramer curve for which $AP = OP$ (Fig. 4.24).

12. Prove that the Strophoid is a special case of the Cramer curve for which the point O coincides with the center P of the circle (Fig. 4.25).

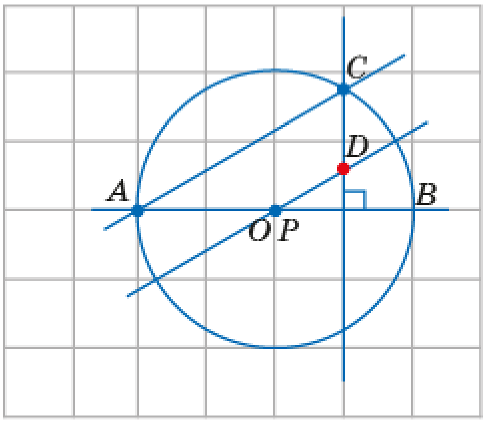


Fig. 4.25

13. Construct the pedal curve of the given circle with respect to the given point A, located on the circle (Fig. 4.26). Obtain this curve using GeoGebra. Prove that this curve is a Pascal snail.

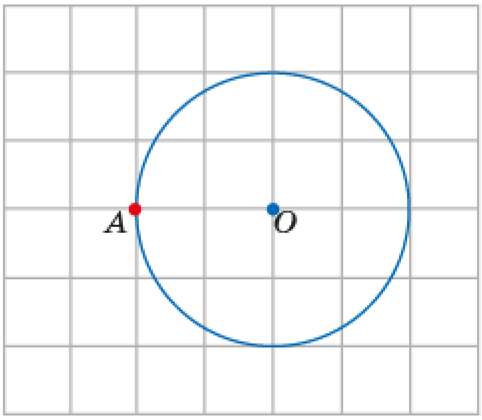


Fig. 4.26

14. Construct the pedal curve of the given circle with respect to the given point A, located inside the circle (Fig. 4.27). Obtain this curve using GeoGebra. Prove that this curve is a Pascal snail.

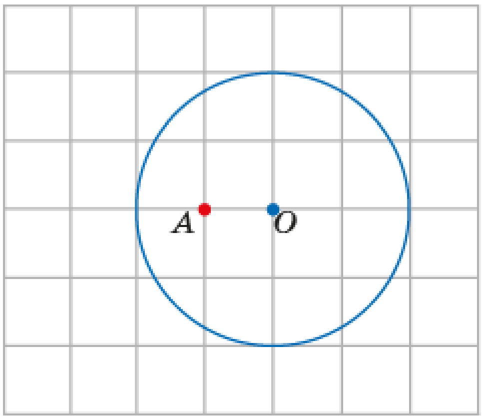


Fig. 4.27

15. Construct the pedal curve of the given parabola with respect to the given point A located on the axis of the parabola, for which the distance to the focus is greater than the distance to the directrix (Fig. 4.28). Obtain this curve using GeoGebra. Prove that this curve is a Cramer's curve.

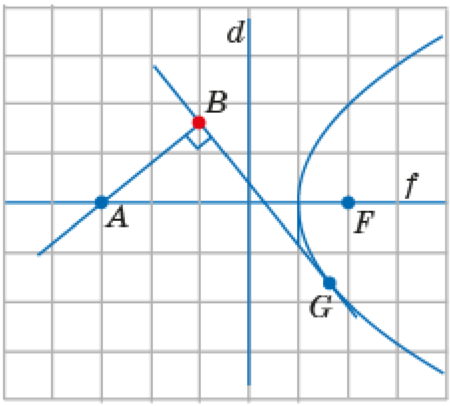


Fig. 4.28

16. Construct the pedal curve of the given parabola with respect to the given point A located at the vertex of the parabola (Fig. 4.29). Obtain this curve using GeoGebra. Prove that this curve is a cissoid of Diocles.

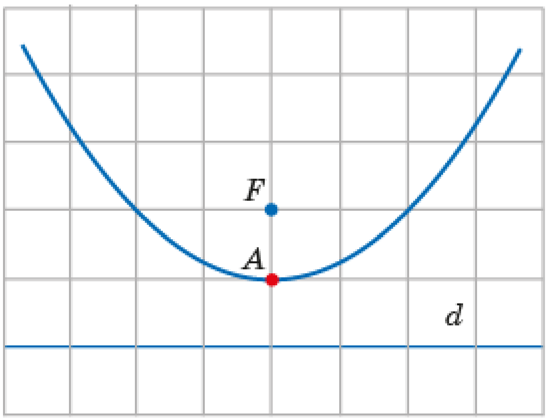


Fig. 4.29

17. Construct the pedal curve of the given parabola with respect to the given point A located at the focus of the parabola (Fig. 4.30). Obtain this curve using GeoGebra. Prove that this curve is a straight line.

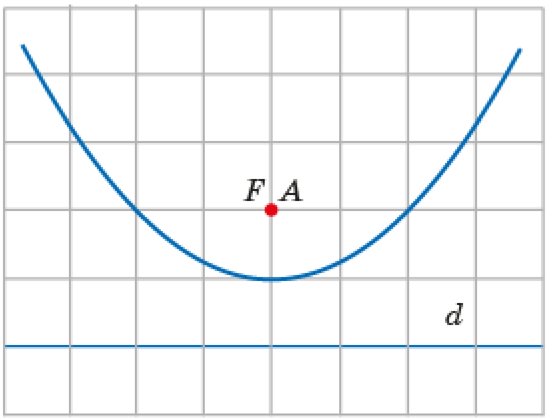


Fig. 4.30

18. Prove that the pedal curve of a given ellipse with foci F_1, F_2 and constant c with respect to the focus F_1 is a circle (Fig. 4.31). Find its center and radius.

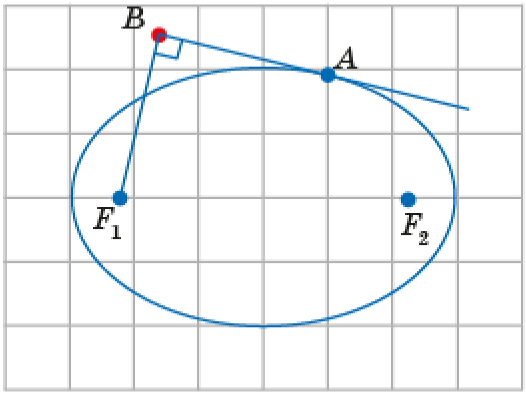


Fig. 4.31

19. Prove that the pedal curve of a given branch of hyperbola with foci F_1, F_2 and constant c with respect to the focus F_1 is a circle (Fig. 4.32). Find its center and radius.

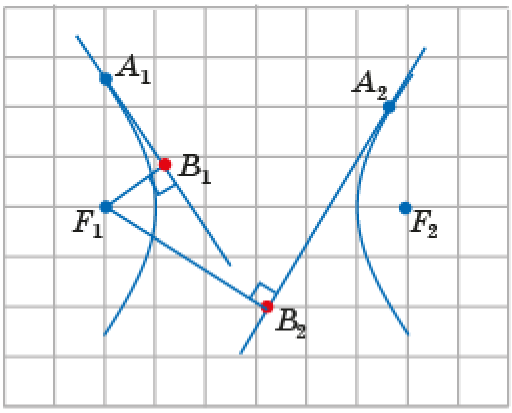


Fig. 4.32

MODULE 2. CURVES AS TRAJECTORUES OF POINTS

5. Cycloid

Consider a circle of radius R rolling along a straight line a. Let C be a point fixed on the circle, and at the initial moment it is at position A (Fig. 5.1). Let us find the trajectory of this point.

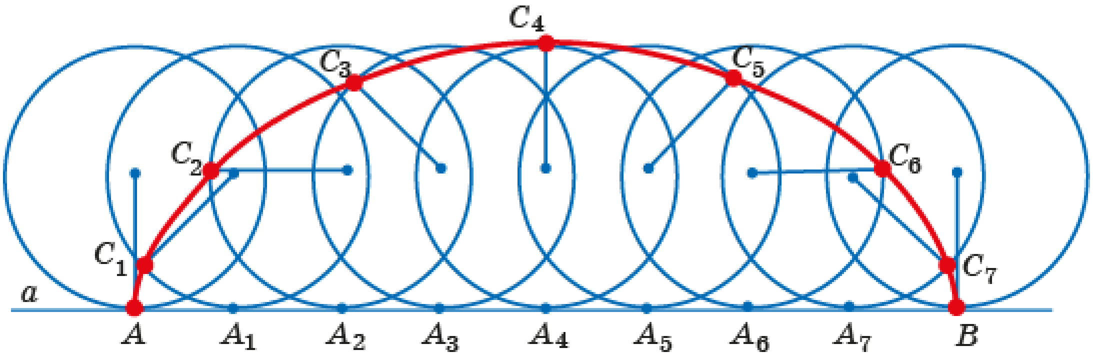


Fig. 5.1

On line a, lay off segment AB equal to the circumference of the circle, i.e., $AB = 2\pi R$. Divide this segment into 8 equal parts by points A_1, A_2, ..., A_7.

If the circle, rolling along line a, moves to position B, it will make one full revolution, i.e., rotate by 360°, and point C will move to position B.

If the circle, rolling along line a, moves to position A_4, it will make half a full revolution, i. e., rotate by 180°, and point C will move to the topmost position C_4.

If the circle, rolling along line a, moves to position A_1, it will rotate by 45°, and point C will move to position C_1.

Figure 5.1 also shows other positions of the circle.

Connecting the constructed points with a smooth curve, we obtain the curve shown in Figure 5.1.

The desired trajectory will consist of a periodically repeating section of the constructed curve.

The curve obtained as the trajectory of a point fixed on a circle rolling along a straight line is called a ***cycloid***.

Such a curve is described, for example, by a point on the rim of a bicycle wheel rolling on a flat road.

The name "cycloid" was coined by Galileo Galilei (1564–1642). Translated from Greek, it means circular, reminiscent of a circle.

A cycloid can be obtained in the GeoGebra computer program. To do this, follow these steps:

Create a "Slider" a, varying from –2 to 8.

In the "Input" line, type: O=(a,1) and press "Enter".

Type: A=(a,0) and press "Enter".

Using the "Circle with Center and Radius" tool, construct a circle with center O and radius 1.

Using the "Rotate around Point" tool, rotate point *A* around point *O* by angle *a* radians clockwise. Label the resulting point *C*.

Using the "Segment" tool, connect points *O* and *C* with a segment.

Right-click on point *C* and select the "Trace On" option.

Remove unnecessary labels and change the color.

Right-click on the slider and select "Animate".

The circle will roll along the x-axis, and the point fixed on it will trace a cycloid (Fig. 5.2).

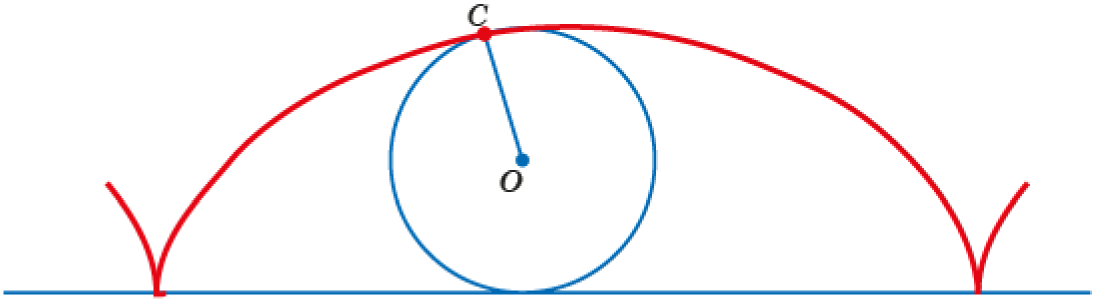


Fig. 5.2

Instead of a circle rolling along a straight line, other figures can be considered.

Consider, for example, an equilateral triangle *ABC* with side length 1, rolling along line *AB*.

Let us determine the trajectory that vertex *A* of this triangle will describe.

First, vertex *A* describes an arc AC_1 of a circle of 120° with center at point A_1 and radius 1, reaching the topmost position C_1.

After that, rotation occurs around point A_2, and point *A* describes an arc C_1B of a circle. Then everything repeats.

Thus, the desired trajectory will consist of two arcs of a unit circle with radius 1 and an angle of 120° (Fig. 5.3).

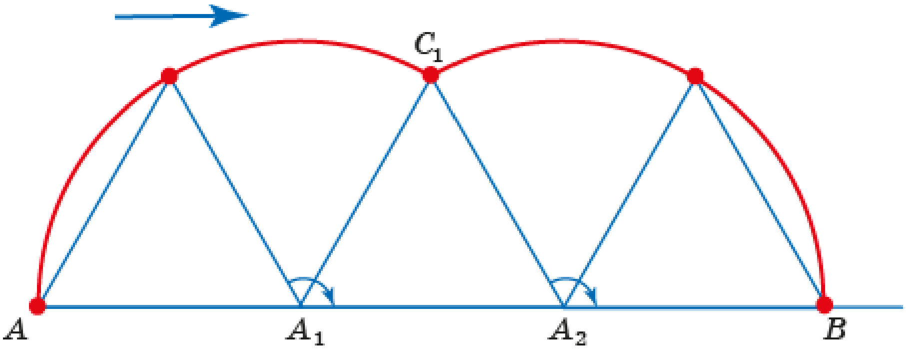


Fig. 5.3

To obtain in GeoGebra the trajectory of a point fixed at the vertex of an equilateral triangle rolling along a straight line, perform the following steps sequentially:

Using the "Slider" tool, create a slider α, varying from 0° to 360°.

Mark points A(0, 0), B(2, 0) and construct an equilateral triangle with vertices A and B.

Mark point D(4, 0) and construct an equilateral triangle with vertices B and D.

Mark point F(6, 0) and construct an equilateral triangle with vertices D and F.

In the "Input" line, type: If(0°≤ α ≤120°, Rotate(Polygon(A, B, 3), –α, B)) and press "Enter".

In the "Input" line, type: If(120°≤ α ≤240°, Rotate(Polygon(B, D, 3), –α + 120°, D)) and press "Enter".

In the "Input" line, type: If(240°≤ α ≤360°, Rotate(Polygon(D, F, 3), –α + 240°, F)) and press "Enter".

In the "Input" line, type: If(0° ≤ α ≤ 120°, Rotate(A, –α, B)) and press "Enter".

In the "Input" line, type: If(120° ≤ α ≤ 240°, Rotate(E, –α + 120°, D)) and press "Enter".

Right-click on the slider and select "Animate". The triangle will roll along the x-axis, and the point fixed on it will trace a curve (Fig. 5.4).

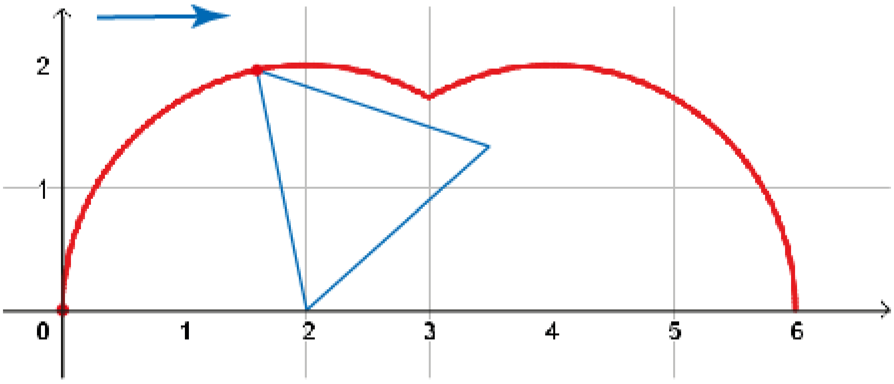


Fig. 5.4

As additional literature dedicated to the cycloid, we recommend the book: Berman, G. N. *The Cycloid*. Moscow: Nauka, 1975.

Exercises

1. Draw the trajectory of a point fixed on a circle of radius 2 cm rolling along a straight line.
2. Does a cycloid have: a) axes of symmetry; b) a center of symmetry?
3. A circle of radius 2 cm rolls along a straight line. At what distances from this line will points C_1, ..., C_7 be located (Fig. 5.1)?
4. Do a lab experiment. Cut a rectangular strip of thick cardboard about 15–20 cm long and a circle of radius 2 cm. Cut a small notch on the edge of the circle into which you can place the tip of a pencil. Glue the strip to a sheet of paper and position the circle so that the cut-out notch is on the edge of the strip. Roll the circle

along the edge of the strip while keeping the pencil tip in the notch. As a result, you will obtain a cycloid (Fig. 5.5).

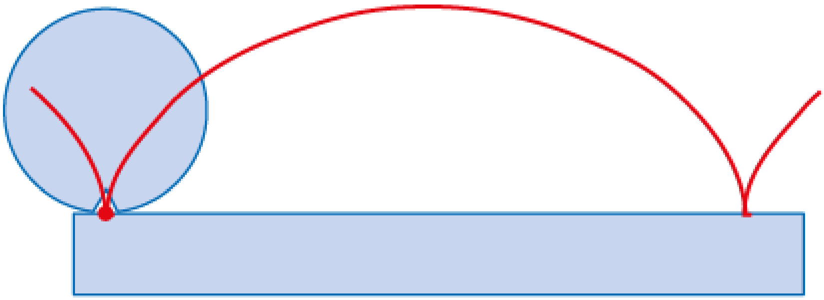


Fig. 5.5

5. Obtain a cycloid using the GeoGebra computer program.

6. Draw the curve that point *C*, fixed on the radius of a circle rolling along a straight line, describes (Fig. 5.6). This curve is called a ***curtate cycloid***.

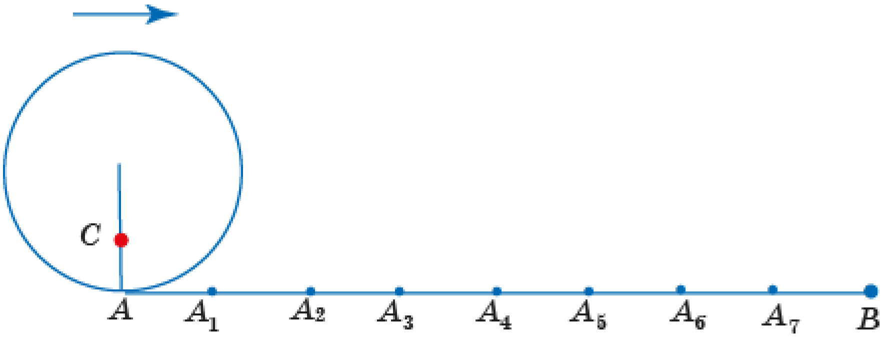


Fig. 5.6

7. Obtain a curtate cycloid using the GeoGebra computer program.

8. Draw the curve that point *C*, fixed on the extension of the radius of a circle rolling along a straight line, describes (Fig. 5.7). This curve is called a ***prolate cycloid***.

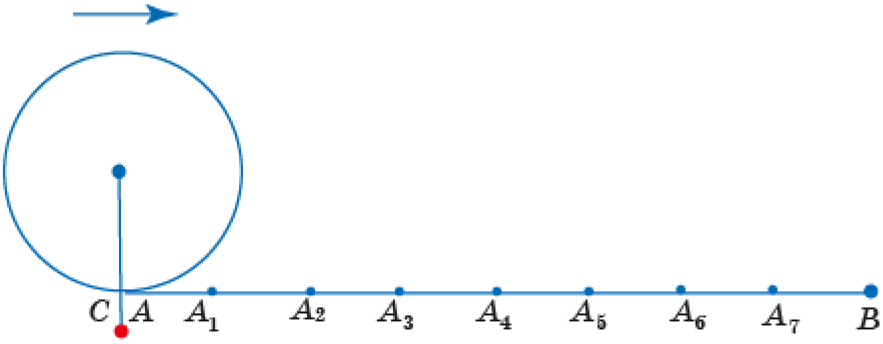


Fig. 5.7

9. Obtain a prolate cycloid using the GeoGebra computer program.

10. Do a lab experiment similar to the previous one, using an equilateral triangle instead of a circle (Fig. 5.8).

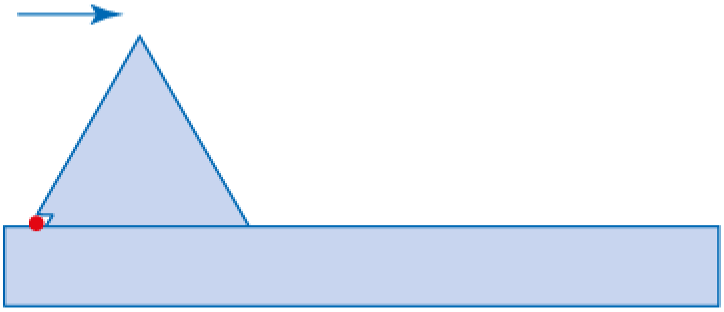


Fig. 5.8

11. Using the GeoGebra computer program, obtain the trajectory of a vertex of an equilateral triangle rolling along a straight line (Fig. 5.4).

12. Draw the trajectory of a vertex of a square rolling along a straight line (Fig. 5.9).

Fig. 5.9

13. Do a lab experiment similar to the previous one, using a square instead of a circle (Fig. 5.10).

Fig. 5.10

14. Using the GeoGebra computer program, obtain the trajectory of a vertex of a square rolling along a straight line.

15. Draw the trajectory of a vertex of a regular hexagon rolling along a straight line (Fig. 5.11).

Fig. 5.11

16. Do a lab experiment similar to the previous one, using a regular hexagon instead of a circle (Fig. 5.12).

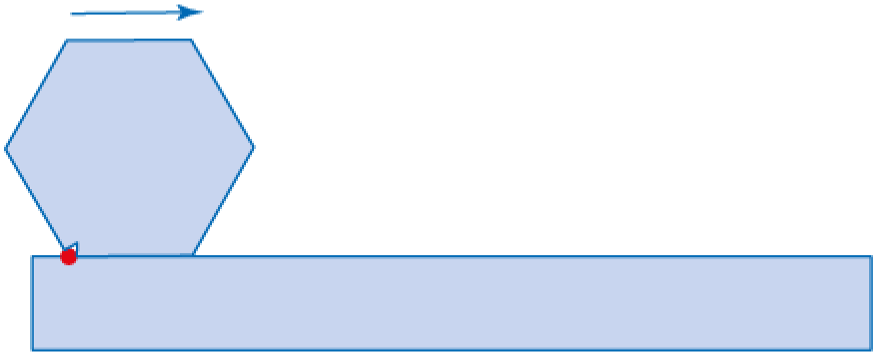


Fig. 5.12

17. Obtain the trajectory of a vertex of a regular hexagon using the GeoGebra computer program.

18. Given a circle and a point *P* located inside it. A point *A* moves along the circle (Fig. 5.13). What trajectory will the midpoint *B* of segment *AP* trace?

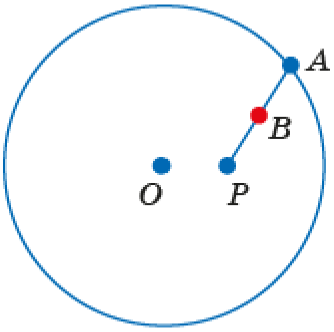


Fig. 5.13

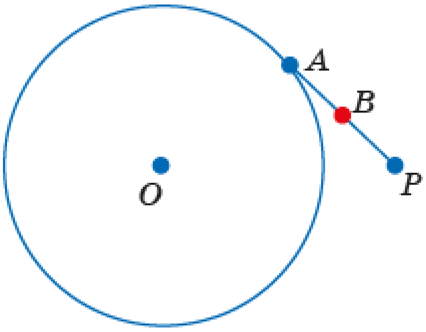


Fig. 5.14

19. Given a circle and a point *P* located outside it. A point *A* moves along the circle (Fig. 5.14). What trajectory will the midpoint *B* of segment *AP* trace?

20. Consider the following problem from the book by Vasiliev N. B., Gutenmacher V. L. "Straight Lines and Curves". M. Nauka, 1978.

A ladder *AB*, standing on a smooth floor against a wall, slides down. Along what line does kitten *K*, sitting in the middle of the ladder, move (Fig. 5.15)?

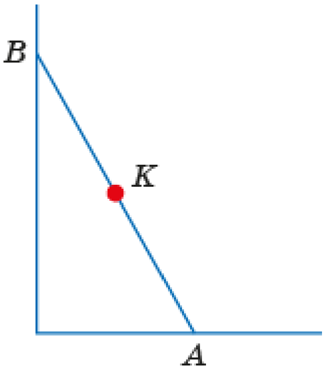


Fig. 5.15

6. Epicycloids and Hypocycloids

Let us now consider the case where a circle rolls not along a straight line, but along another circle on its outer side.

Depending on the ratio of the radii of the fixed and rolling circles, various curves are obtained. They are called ***epicycloids***.

The curve obtained as the trajectory of a point fixed on a circle rolling externally along another circle of the same radius is called a ***cardioid***.

Let us construct some points of this curve. Suppose point C, fixed on the circle, is initially at position A (Fig. 6.1). Divide the fixed circle into 8 equal parts by points A_1, A_2, ..., A_7.

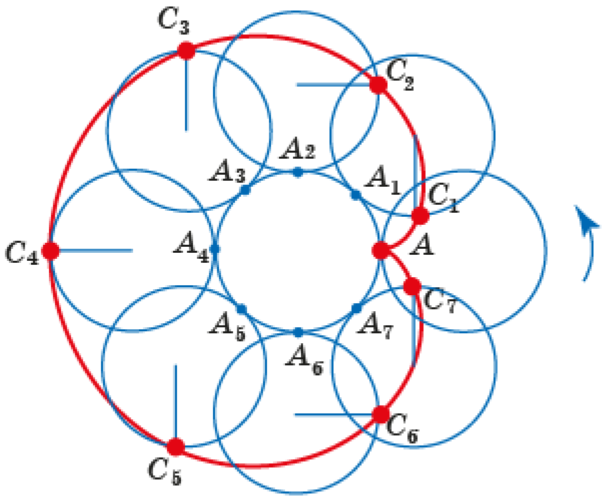


Fig. 6.1

If the circle moves to position A_4, it will have made half a full revolution, i.e., it will have rotated by 180°, and point A will move to position C_4.

If the circle moves to position A_1, it will have rotated by 45°, and point A will move to position C_1.

Figure 6.1 also shows other points of the cardioid corresponding to the remaining positions of the rolling circle.

Connect the constructed points with a smooth curve. The result is the desired cardioid.

A cardioid can be obtained using the GeoGebra computer program. To do this, create a slider α, varying from 0° to 360°.

In the Input bar, type O=(0,0) and press Enter. This gives point $O(0, 0)$.

Using the "Circle with Center and Radius" tool, create a circle with center O and radius 1.

In the Input bar, type P=(2,0) and press Enter. This gives point $P(2, 0)$.

Using the "Rotate around Point" tool, rotate point P around point O by angle α counterclockwise. This gives point Q.

Using the "Circle with Center and Radius" tool, create a circle with center Q and radius 1.

In the Input bar, type A=(1,0) and press Enter. This gives point $A(1, 0)$.

Using the "Rotate around Point" tool, rotate point A around point O by angle α counterclockwise. This gives point B.

Using the "Rotate around Point" tool, rotate point B around point Q by angle α counterclockwise. This gives point C.

Using the "Segment" tool, connect points Q and C with a segment.

Left-click on point C and select "Trace On".

Left-click on the slider and select "Animate". As a result, the circle will roll along the circle with center O, and point C will trace a cardioid (Fig. 6.2).

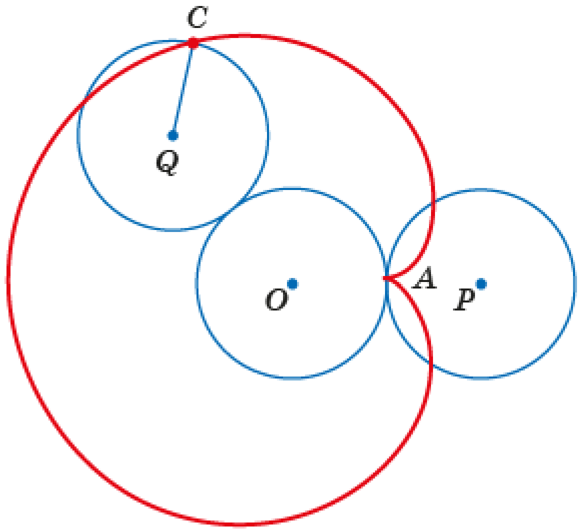


Fig. 6.2

If the radius of the rolling circle is three times smaller than the radius of the fixed circle, the curve shown in Figure 6.3 is obtained.

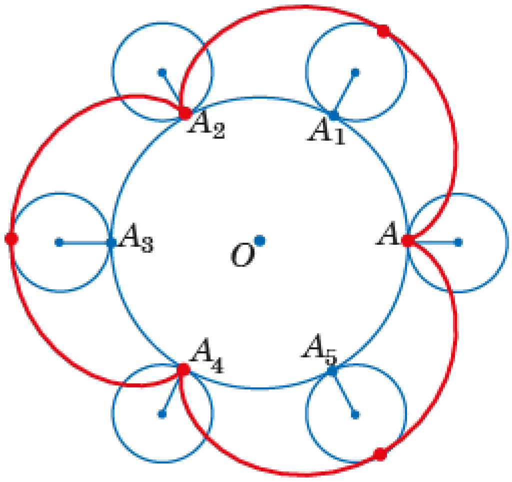


Fig. 6.3

If the radius of the rolling circle is twice the radius of the fixed circle, the curve shown in Figure 6.4 is obtained.

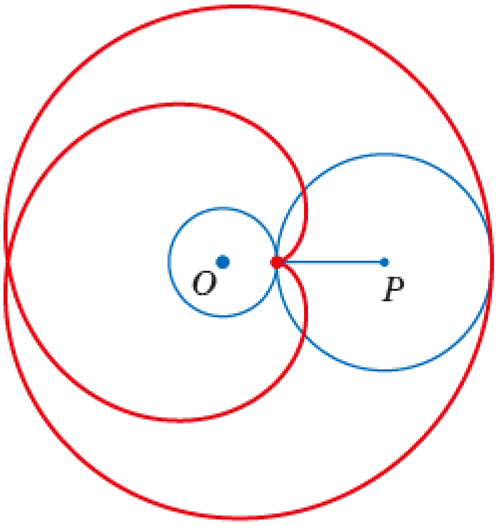


Fig. 6.4

Instead of a circle rolling along another circle, one can also consider regular polygons rolling along another regular polygon.

Consider, for example, a regular triangle rolling along an equal regular triangle *ABC* (Fig. 6.5).

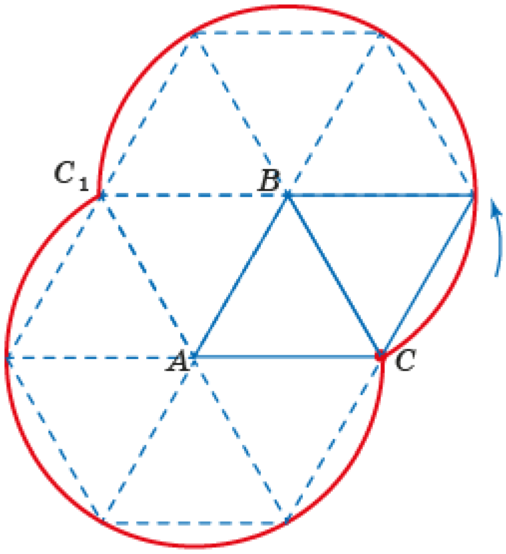


Fig. 6.5

First, vertex *C* describes an arc of a circle of 240° with center at *B* and radius equal to the side of the triangle. As a result of this rotation, point *C* moves to point C_1.

After this, rotation occurs around vertex *A*, and point C_1 describes an arc of a circle of 240° with center at *A* and radius equal to the side of the triangle.

As a result of this rotation, point C_1 moves to point *C*. The process then repeats.

Thus, the desired trajectory consists of two arcs of a circle of 240° with radius equal to the side of the triangle (Fig. 6.5).

Now consider the case where a circle rolls along another circle on its inner side. Depending on the ratio of the radii of the fixed and rolling circles, various curves are obtained. They are called ***hypocycloids***.

A special case of such a curve is the ***astroid*** — the trajectory of a point fixed on a circle rolling internally along another circle four times larger in radius (Fig. 6.6).

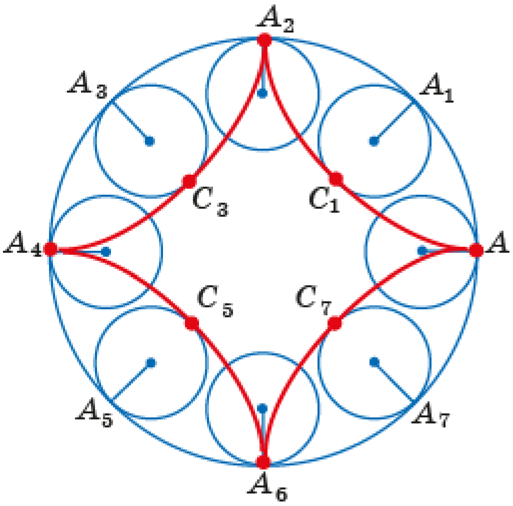


Fig. 6.6

Exercises

1. Draw the trajectory of a point fixed on a circle of radius 2 cm rolling along a circle of the same radius.

2. Does a cardioid have: a) axes of symmetry; b) a center of symmetry?

3. Perform a laboratory activity. Cut out two circles of radius 2 cm from thick cardboard. Glue one circle to a sheet of paper, and cut a small notch on the edge of the second circle into which you can place a pencil point. Place the second circle so that the cut notch is on the edge of the first. Roll the second circle along the first circle while keeping the pencil point in the notch. As a result, you will obtain a cardioid (Fig. 6.7).

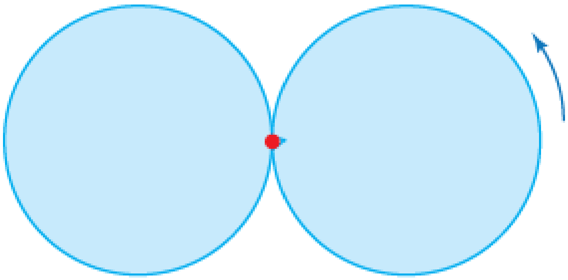


Fig. 6.7

4. Obtain a cardioid using the GeoGebra computer program.

5. Prove that the cardioid is a special case of the limaçon of Pascal.

6. Draw the curve that point C, fixed on the radius of a circle rolling along another circle of the same radius, will trace (Fig. 6.8). This curve is called a ***curtate cardioid***.

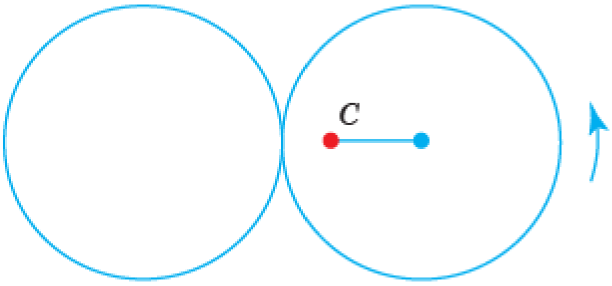


Fig. 6.8

7. Obtain the curtate cardioid using the GeoGebra computer program.

8. Prove that the curtate cardioid is a special case of the limaçon of Pascal.

9. Draw the curve that point C, fixed on the extension of the radius of a circle rolling along another circle of the same radius, will trace (Fig. 6.9). This curve is called a ***prolate cardioid***.

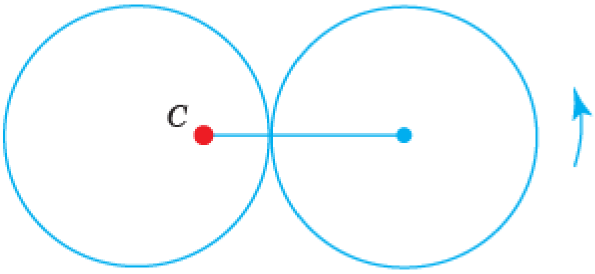


Fig. 6.9

10. Obtain the prolate cardioid using the GeoGebra computer program.

11. Prove that the prolate cardioid is a special case of the limaçon of Pascal.

12. Using the GeoGebra computer program, obtain the curve traced by a point fixed on a circle of radius 1 cm rolling externally along a circle of radius 3 cm.

13. Draw the curve that point C, fixed on a circle of radius 2 cm rolling externally along another circle of radius 5 cm, will trace. Obtain this curve using the GeoGebra computer program.

14. Obtain an astroid using the GeoGebra computer program.

15. Draw the curve that point C, fixed on a circle of radius 1 cm rolling internally along another circle of radius 3 cm, will trace. It is called a *Steiner curve*. Obtain it using the GeoGebra computer program.

16. Draw the curve that point C, fixed on a circle of radius 2 cm rolling internally along another circle of radius 5 cm, will trace. Obtain it using the GeoGebra computer program.

17. Draw the curve that point C, fixed on a circle of radius 2 cm rolling internally along another circle of radius 1 cm, will trace (Fig. 6.10). Obtain it using the GeoGebra computer program. Determine what this curve is. Prove it.

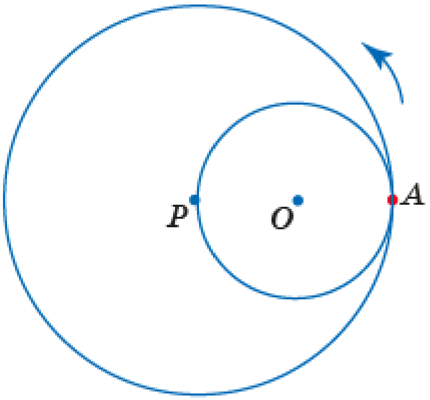


Fig. 6.10

18. Prove that the trajectory of a point fixed on a circle of radius R, rolling internally along a given circle with center O and radius 1 (R > 1), coincides with the trajectory of a point fixed on a circle of radius R – 1, rolling externally along the same given circle.

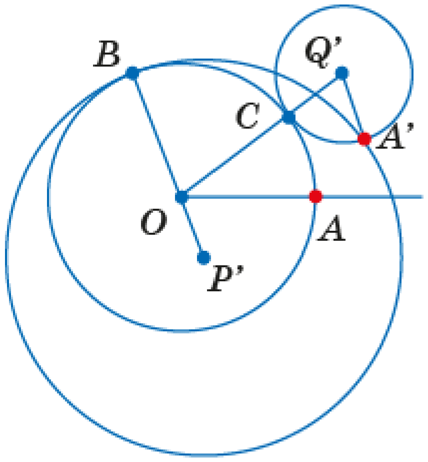


Fig. 6.11

19. Perform a laboratory activity. Cut out two equal regular triangles from thick cardboard. Glue one triangle to a sheet of paper, and cut a small notch near the vertex of the second triangle into which you can place a pencil point. Place the second triangle on the side of the first so that the cut notch is at the vertex of the first. Rotate the second triangle around the vertex of the first while keeping the pencil point in the notch. As a result, you will obtain the desired trajectory (Fig. 6.12).

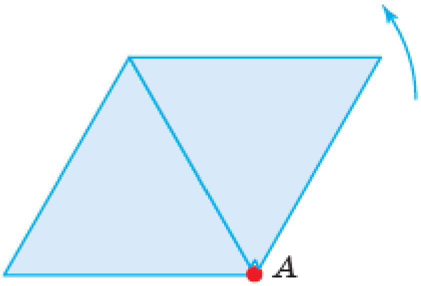


Fig. 6.12

20. Draw the trajectory of the vertex of a square rolling along another square (Fig. 6.13).

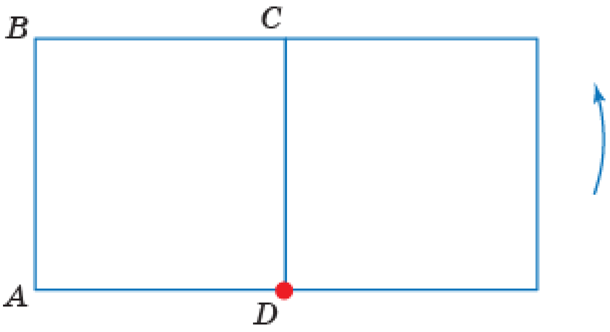


Fig. 6.13

21. Perform a laboratory activity similar to the previous one, using squares instead of triangles (Fig. 6.14).

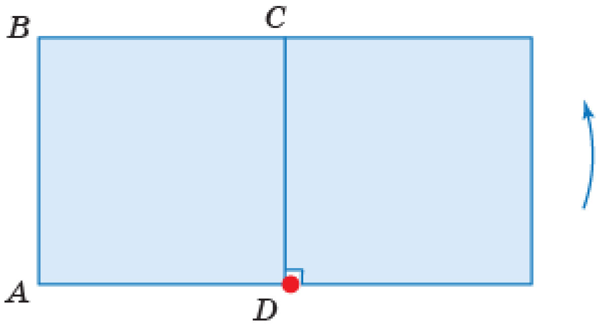


Fig. 6.14

22. Draw the trajectory of the vertex of a regular hexagon rolling along an equal regular hexagon (Fig. 6.15).

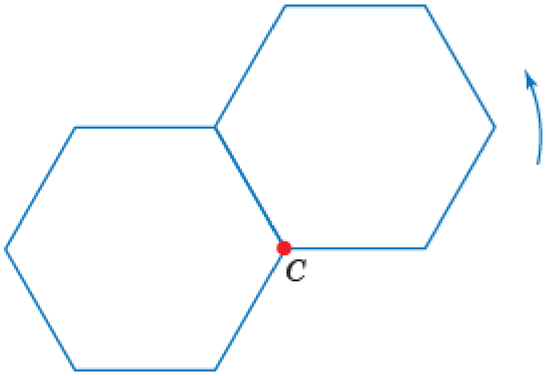


Fig. 6.15

23. Perform a laboratory activity similar to the previous one, using regular hexagons instead of squares (Fig. 6.16).

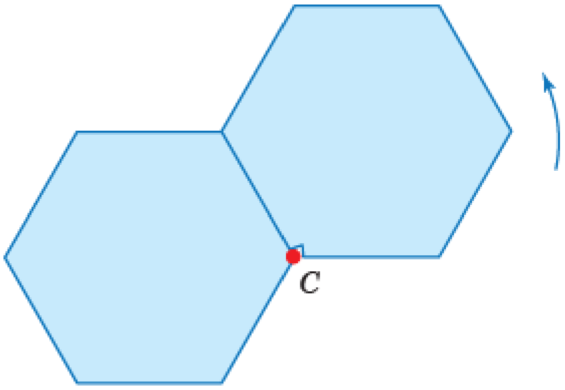


Fig. 6.16

24. A unit regular triangle rolls inside a unit square (Fig. 6.17). Draw the trajectory of a vertex of this triangle.

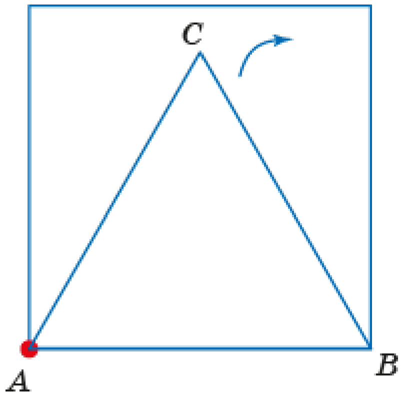


Fig. 6.17

25. A unit square rolls inside a unit regular hexagon (Fig. 6.18). Draw the trajectory of a vertex of this square.

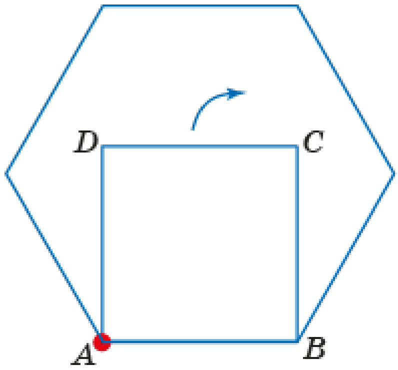


Fig. 6.18

7. Envelopes

Let's consider another method of forming curves, using the concept of an envelope.

A curve is called an ***envelope*** of a family of lines if each line of this family is tangent to this curve, and each point of this curve is the point of tangency of one of the lines of this family.

For example, a circle centered at the origin with radius 1 can be considered as the envelope of a family of lines obtained by all possible rotations of the line $x = 1$ about the origin.

To model this family of lines in the GeoGebra computer program, mark point $O(0, 0)$ and draw the line $x = 1$. Create a slider t, varying from 0 to 2π. Using the "Rotate around Point" tool, rotate the constructed line around point O by angle t counterclockwise. In the properties of this line, select the "Show trace" option. By changing the slider value from 0 to 2π, we obtain the desired family of lines, whose envelope is this circle (Fig. 7.1).

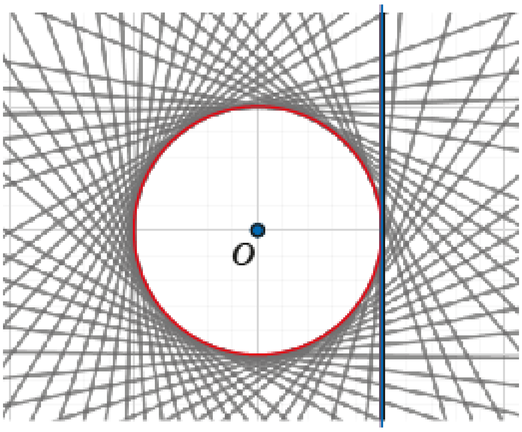


Fig. 7.1

Let us give other examples of curves that are envelopes of families of lines.

Parabola. As shown in section 1, the perpendicular bisectors c of the segments FD connecting the focus F of the parabola with points D on the directrix d are tangents to the parabola. Consequently, the parabola with focus F and directrix d is the envelope of the family of perpendicular bisectors of all possible segments connecting point F with points on the line d (Fig. 7.2).

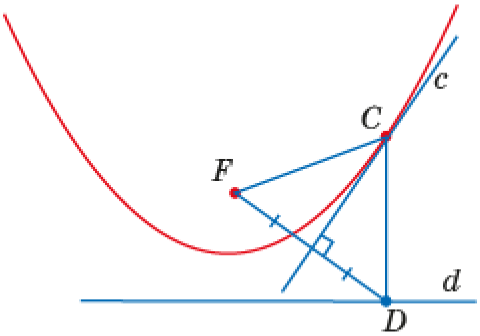


Fig. 7.2

To model this family of lines, we use the GeoGebra computer program.

Create a slider a ($0 < a < 5$). Draw the line $y = 0$ and mark point $F(0, a)$. Mark point D on line d. Draw segment FD. Construct its perpendicular bisector. In the properties of this perpendicular bisector, select the "Trace On" option. By moving point D along line d, we obtain the desired family of perpendicular bisectors of segments FD. The envelope of this family will be a parabola with focus F and directrix d (Fig. 7.3).

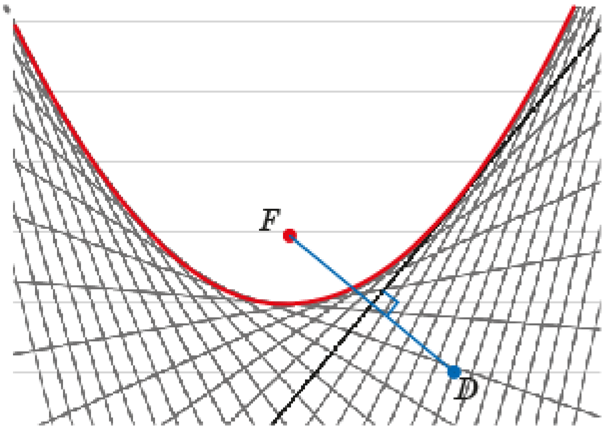


Fig. 7.3

2. Ellipse. Consider a circle centered at point F_1. Mark a point F_2 inside it, distinct from point F_1. As shown in section 2, the perpendicular bisectors c of segments F_2D connecting point F_2 with points D on this circle are tangents to an ellipse whose foci are points F_1 and F_2 (Fig. 7.4). Consequently, the ellipse with foci F_1 and F_2 is the envelope of the family of perpendicular bisectors of all possible segments connecting point F_2 with points on the considered circle.

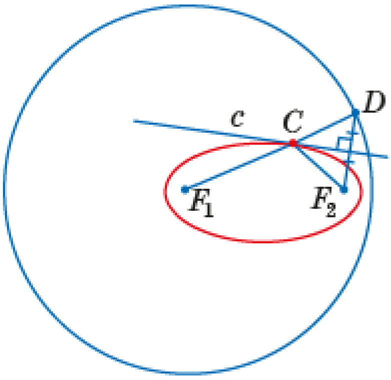


Fig. 7.4

To model this family of lines, we use the GeoGebra computer program.

Draw a circle with center $F_1(0, 0)$ and radius $R = 1$. Create a slider a ($0 < a < R$). Mark point $F_2(a, 0)$ inside the circle. Mark point D on this circle. Draw segment F_2D. Construct the perpendicular bisector of segment F_2D. In the properties of this perpendicular bisector, select the "Trace On" option. By moving point D along the circle, we obtain the desired family of perpendicular bisectors of segments F_2D. The envelope of this family will be an ellipse with foci F_1 and F_2 (Fig. 7.5).

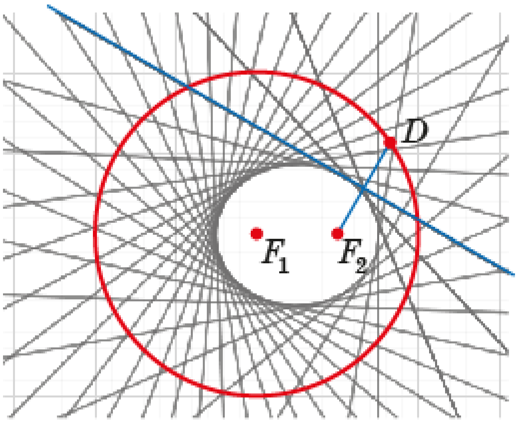


Fig. 7.5

3. Hyperbola. Consider a circle with center at point F_1 and radius R. Mark a point F_2 outside it. As shown in section 3, the perpendicular bisectors c of segments F_2D connecting point F_2 with points D on this circle are tangents to a hyperbola whose foci are points F_1 and F_2 (Fig. 7.6). Consequently, the hyperbola with foci F_1 and F_2 is the envelope of the family of perpendicular bisectors of all possible segments connecting point F_2 with points D on the considered circle.

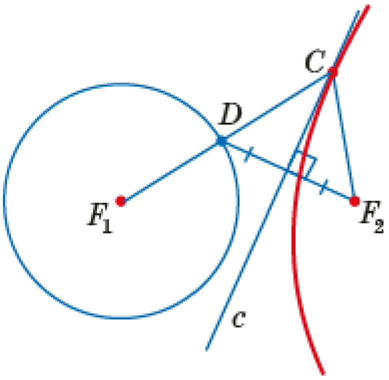


Fig. 7.6

To model this family of lines, we use the GeoGebra computer program.

Draw a circle with center $F_1(0, 0)$ and radius $R = 1$. Create a slider a ($R < a < 2$). Mark point $F_2(a, 0)$ outside the circle. Mark point D on this circle. Draw segment F_2D. Construct the perpendicular bisector of segment F_2D. In the properties of this perpendicular bisector, select the "Trace On" option. By moving point D along the circle, we obtain the desired family of perpendicular bisectors of segments F_2D. The envelope of this family will be a hyperbola with foci F_1 and F_2 (Fig. 7.7).

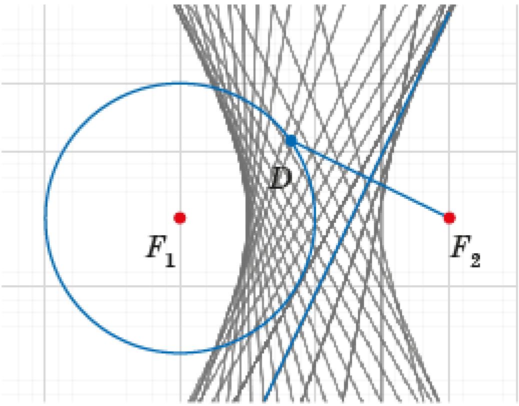


Fig. 7.7

4. Cycloid. Let us define the tangent to a cycloid at its point C as the limiting position of the secant line CD as point D on the cycloid approaches point C (Fig. 7.8).

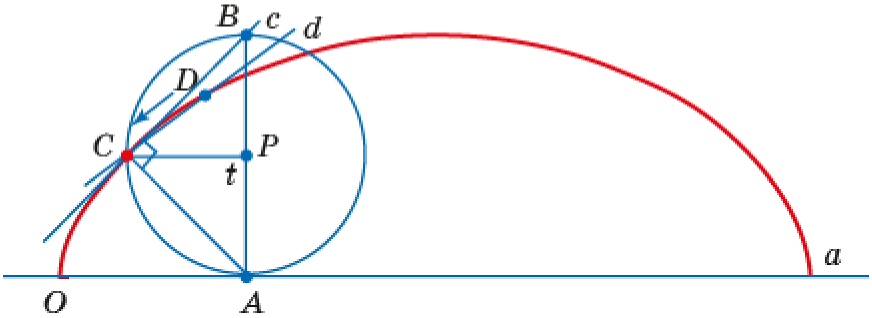


Fig. 7.8

Let us show that the tangent to the cycloid at point C is perpendicular to the segment connecting this point with point A, where the given circle touches the line a along which the circle rolls.

We use the fact that the trajectory of the vertex of a regular polygon rolling along line a consists of circular arcs whose centers lie on line a, and the tangents to these arcs are perpendicular to the radii drawn to the points of tangency. Considering that a circle rolling along a line can be approximated by regular polygons inscribed in this circle, we obtain that the tangents to the cycloid will have this property.

If the central angle APC, through which the rolling circle has turned, equals t, then the tangent c at point C forms an angle with the diameter AB equal to $\frac{t}{2}$. If the radius of the rolling circle is 1, then the magnitude of the central angle APC, measured in radians, equals the length of the segment OA over which the circle has rolled. Consequently, the cycloid is the envelope of a family of lines, each obtained by rotating line AB around point B by angle $\frac{t}{2}$ clockwise.

To model this family of lines, we use the GeoGebra computer program.

Draw a line a given by the equation $y = 0$. Mark point $O(0, 0)$ on it. At a distance of 2 from this line, draw a line b parallel to it, given by the equation $y = 2$. Create a slider t, varying from 0 to 2π. On line a, mark point A such that $OA = t$. Through point A, draw a line perpendicular to line a. Construct point B as the intersection of this line with line *b*. Rotate the constructed line *AB* around point *B* by angle $\frac{t}{2}$ clockwise. In the properties of this line, select the "Trace On" option. By changing the slider value from 0 to 2π, we obtain the desired family of lines, whose envelope is the cycloid (Fig. 7.9).

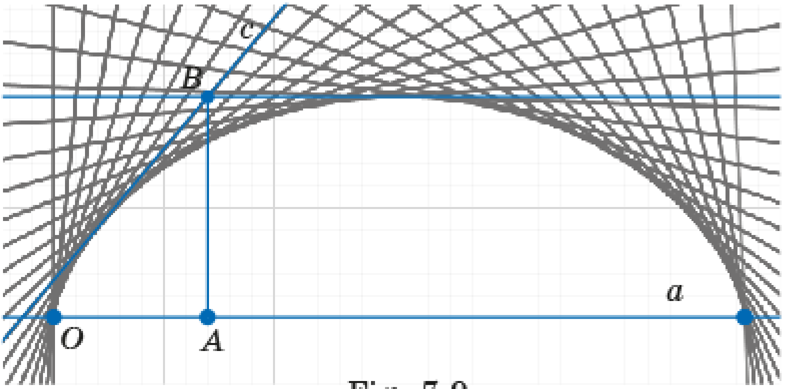


Fig. 7.9

5. Cardioid. Similarly to the definition of the tangent to the cycloid, we define the tangent to the cardioid at its point *C* as the limiting position of the secant line *CD* as point *D* on the cardioid approaches point *C* (Fig. 7.10).

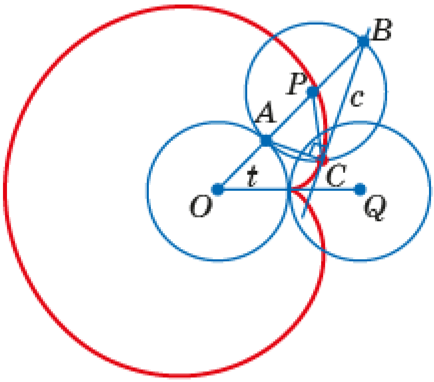


Fig. 7.10

Similarly to what was done for the cycloid, we show that the tangent to the cardioid at point *C* is perpendicular to the segment connecting this point with point *A*, where the rolling circle touches the circle along which it rolls. We use the fact that the trajectory of the vertex of a regular polygon rolling along an equal regular polygon consists of circular arcs, and the tangents to these arcs are perpendicular to the radii drawn to the points of tangency. Considering that circles can be approximated by regular polygons inscribed in these circles, we obtain that the tangents to the cardioid will have this property.

If the angle *POQ*, through which the rolling circle has turned, equals *t*, then the tangent *c* at point *C* forms an angle with line *AB* equal to $\frac{t}{2}$. From this it follows that the cardioid is the envelope of a family of lines, each obtained by rotating line *AB* around point *B* by angle $\frac{t}{2}$ counterclockwise.

To model this family of lines, we use the GeoGebra computer program.

Construct circles with center *O*(0, 0) and radii 1 and 3. Using the "Slider" tool, create a slider *t*, varying from 0 to 2π. Mark point *D*(3, 0). Rotate it around point *O* by angle *t* counterclockwise. Denote the resulting point *B*. Draw line *OB*. Rotate it around point *B* by angle $\frac{t}{2}$ counterclockwise. In the properties of the resulting line, select the "Trace On" option. By changing the slider value from 0 to 2π, we obtain the desired family of lines, whose envelope is the cardioid (Fig. 7.11).

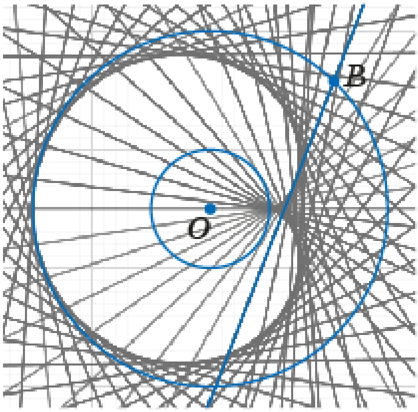


Fig. 7.11

6. Astroid. Define the tangent to the astroid at its point *C* as the limiting position of the secant line *CD* as point *D* on the astroid approaches point *C* (Fig. 7.12).

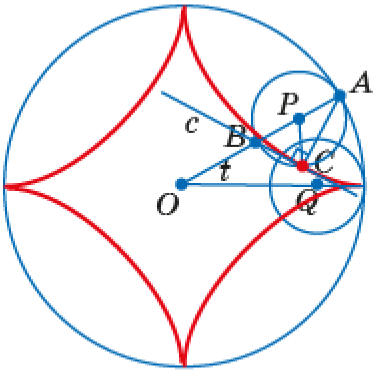


Fig. 7.12

As for the cardioid, the tangent *c* to the astroid at point *C* is perpendicular to the segment connecting this point with point *A*, where the rolling circle touches the circle along which it rolls. If the central angle *POQ* equals *t*, then the tangent *c* at point *C* forms an angle with line *AB* equal to 2*t*. From this it follows that the astroid is the envelope of a family of lines, each obtained by rotating line *AB* around point *B* by angle 2*t* clockwise.

To model this family of lines, we use the GeoGebra computer program.

Construct circles with center *O*(0, 0) and radii 2 and 4. Create a slider *t*, varying from 0 to 2π. Mark point *D*(2, 0). Rotate it around point *O* by angle *t* counterclockwise. Denote the resulting point *B*. Draw line *OB*. Rotate it around point *B* by angle 2*t* clockwise. In the properties of the resulting line, select the "Trace On" option. By changing the slider value from 0 to 2π, we obtain the desired family of lines, whose envelope is the astroid (Fig. 7.13).

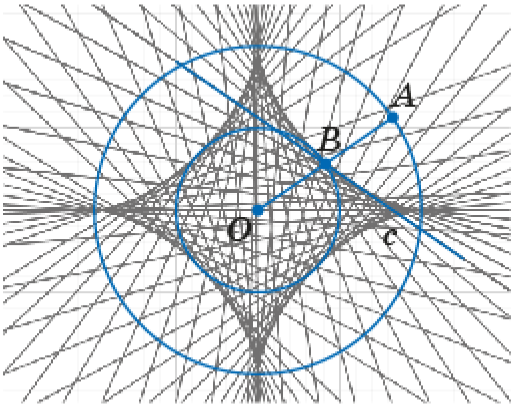


Fig. 7.13

As additional literature on envelopes, we recommend the book: Boltyansky, V. G. *Envelope*. -- Moscow: Fizmatlit, 1961.

Exercises

1. Given a point A and a line b not passing through this point. For given points B on line b, draw lines c perpendicular to segment AB (Fig. 7.14). For all possible positions of point B on line b, find the envelope of this family of lines.

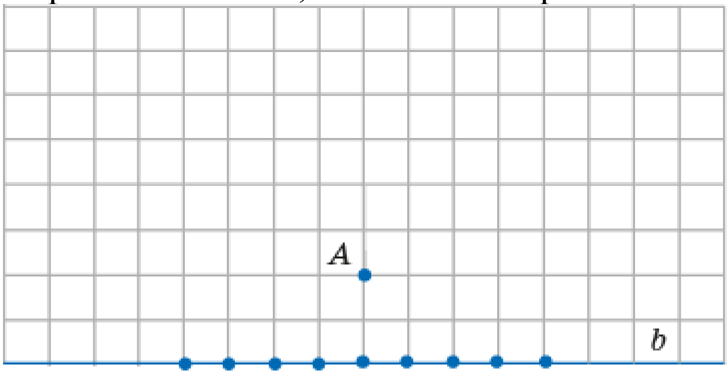


Fig. 7.14

2. Given a circle b with center O and a point A located inside this circle, distinct from the center O. For given points B on this circle, lines c are drawn perpendicular to segment AB (Fig. 7.15). For all possible positions of point B on this circle, find the envelope of this family of lines.

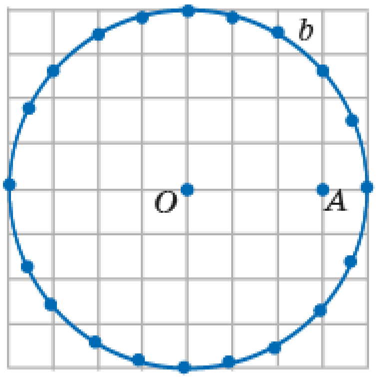


Fig. 7.15

3. Given a circle *b* with center *O* and a point *A* located outside this circle. For points *B* on this circle, lines *c* are drawn perpendicular to segment *AB* (Fig. 7.16). For all possible positions of point *B* on this circle, find the envelope of this family of lines.

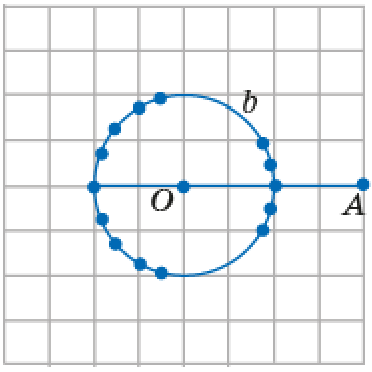


Fig. 7.16

4. Point *C* is fixed on a circle rolling along a line. Construct the family of diameters *CD* of this circle, one of whose endpoints is this point. Find the envelope of this family (Fig. 7.17).

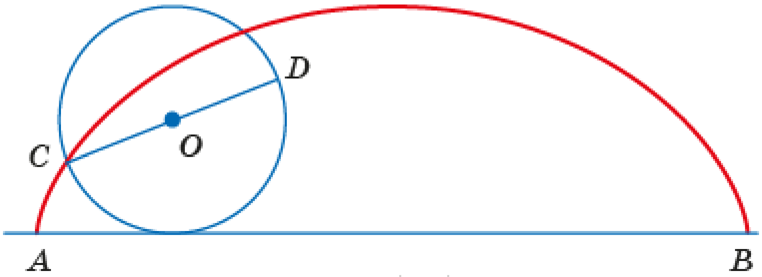


Fig. 7.17

5. Given a line *a* and a point *A* belonging to this line. Point *B* belongs to line *a* and is located at a distance *t* $(0 \leq t \leq 2\pi)$. from point *A*. Line *b* passes through point *B* and is perpendicular to line *a*. Line *c* is obtained by rotating line *b* around point *B* by angle $\frac{t}{2}$ clockwise. Construct the family of such lines *c* for *t* varying from 0 to 2π. Find the envelope of this family (Fig. 7.18).

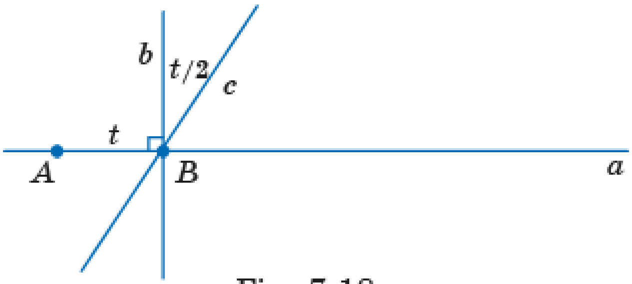


Fig. 7.18

6. Given a circle with center *O*. Point *A* belongs to this circle. Point *B* is obtained by rotating point *A* around point *O* by angle *t* counterclockwise $(0 \leq t \leq 2\pi)$. Line *c* is obtained by rotating line *OB* around point *B* by angle $\frac{t}{2}$

counterclockwise (Fig. 7.19). Construct the family of such lines c for angles t varying from 0 to 2π. Find the envelope of this family.

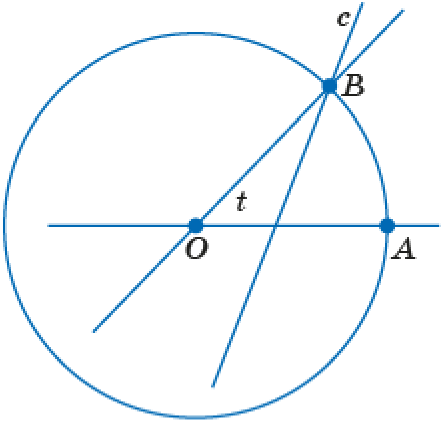


Fig. 7.19

7. Given a circle with center O. Point A belongs to this circle. Point B is obtained by rotating point A around point O by angle t counterclockwise ($0 \le t \le 2\pi$). Line c is obtained by rotating line OB around point B by angle $2t$ counterclockwise (Fig. 7.20). Construct the family of such lines c for angles t varying from 0 to 2π. Find the envelope of this family.

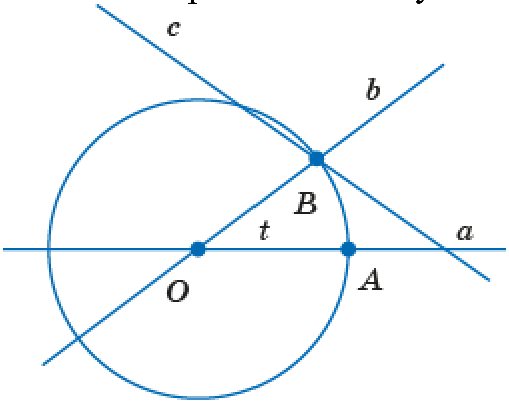


Fig. 7.20

8. Point C is fixed on a circle of radius 2 rolling externally along another circle of radius 1. Construct the family of diameters CD of the rolling circle, one of whose endpoints is this point. Find the envelope of this family (Fig. 7.21).

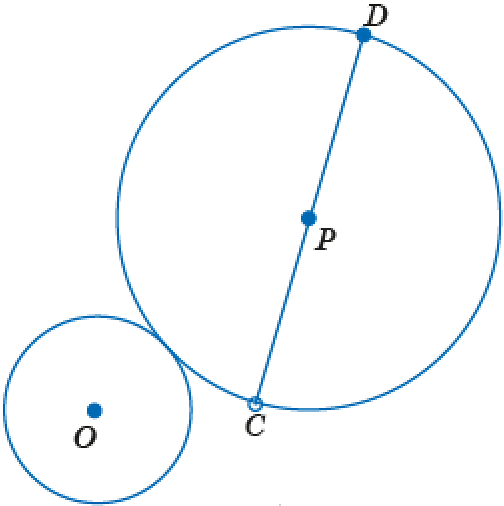


Fig. 7.21

9. Point C is fixed on a unit circle rolling along another unit circle. Construct the family of diameters CD of the rolling circle, one of whose endpoints is this point. Find the envelope of this family (Fig. 7.22).

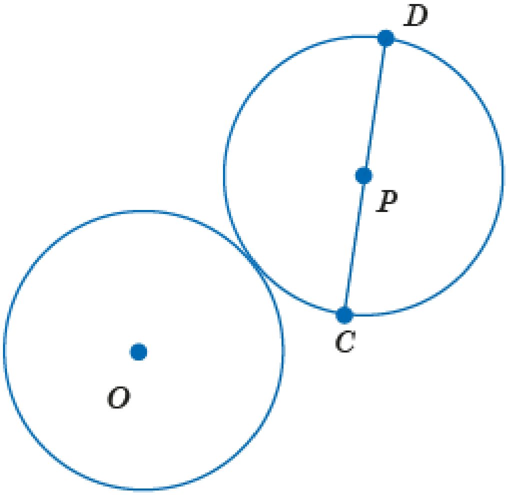


Fig. 7.22

10. A point is fixed on a unit circle rolling inside another circle of radius 2. Find the envelope of the family of diameters of the rolling circle, one of whose endpoints is this point (Fig. 7.23).

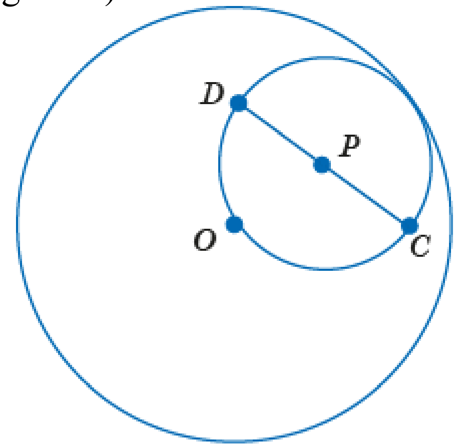


Fig. 7.23

11. A point is fixed on a unit circle rolling inside another circle of radius 1.5 times the radius. Find the envelope of the family of diameters of the rolling circle, one of whose endpoints is this point (Fig. 7.24).

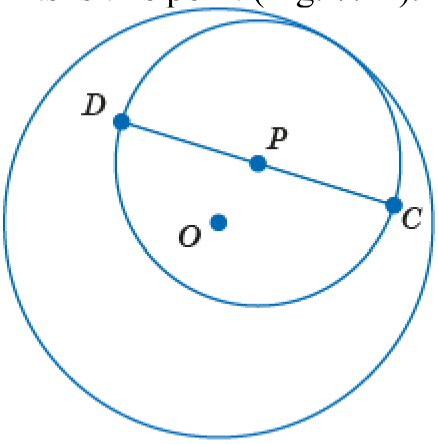


Fig. 7.24

12. For a given right angle aCb, find the envelope of the family of line segments AB of constant length $c = 8$, where the endpoints A and B lie on the sides a and b of the given angle, respectively (Fig. 7.25).

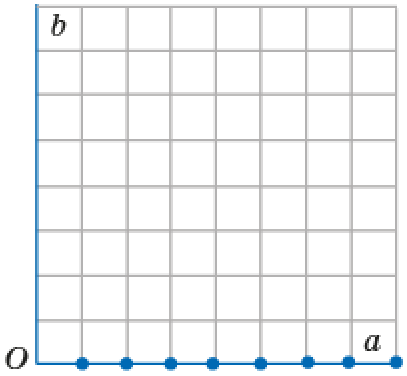


Fig. 7.25

13. For a given right angle aOb, find the envelope of the family of line segments AB, where the endpoints A and B belong to the sides a and b of the given angle, respectively, and the sum of the lengths of segments OA and OB is constant and equal to $c = 8$ (Fig. 7.26). Prove it for any angle and $c > 0$.

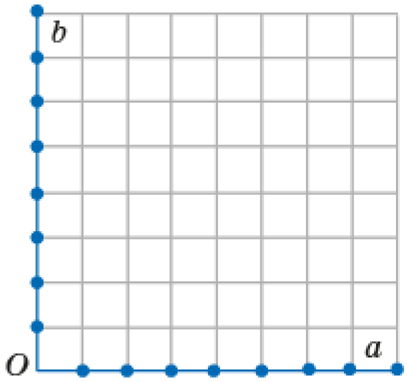


Fig. 7.26

14. For a given right angle aOb and a point C located inside it (Fig. 7.27) find the envelope of the family of segments AB, whose endpoints belong to the sides of the given angle, and which are visible from point C under the given angle $\angle ACB = 90°$. Prove it for any angle aOb and ACB such that $\angle aOb + \angle ACB = 180°$.

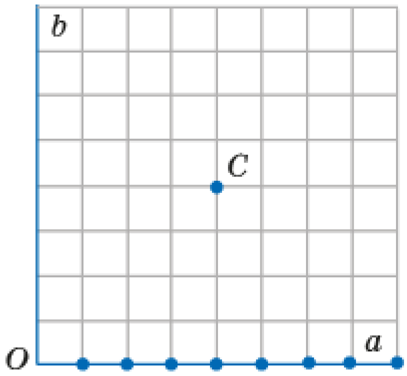


Fig. 7.27

15. For a given right angle aOb and a point C located inside it (Fig. 7.28) find the envelope of the family of segments AB, whose endpoints belong to the sides of the given angle, and which are visible from point C under the given angle $\angle ACB = 60°$. Prove it for any angle aOb and ACB such that $\angle aOb + \angle ACB < 180°$.

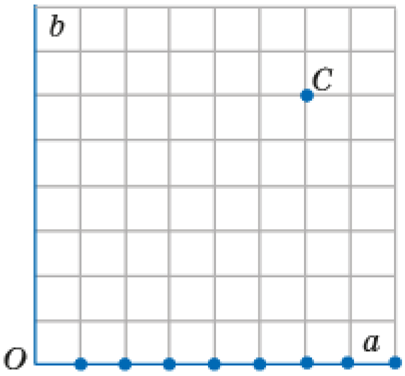


Fig. 7.28

16. For a given right angle aOb and a point C located inside it (Fig. 7.29) find the envelope of the family of segments AB, whose endpoints belong to the sides of the given angle, and which are visible from point C under the given angle $\angle ACB = 120°$. Prove it for any angle aOb and ACB such that $\angle aOb + \angle ACB > 180°$.

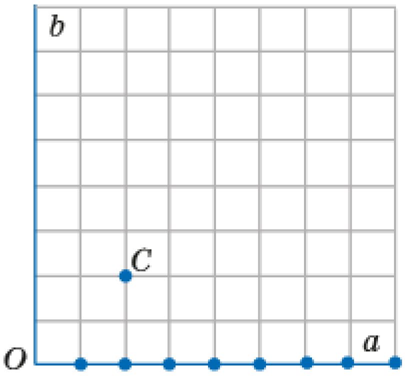


Fig. 7.29

17. Given a circle with center O and a point A on this circle. Find the envelope of the family of circles passing through point A, whose centers P lie on the given circle (Fig. 7.30).

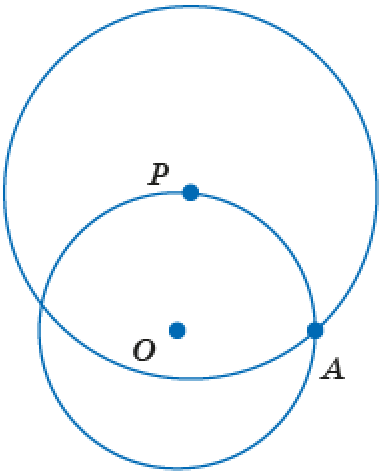


Fig. 7.30

18. Given a circle with center O and a point A on this circle. For all possible points B on this circle, find the envelope of the family of circles with diameters AB (Fig. 7.31).

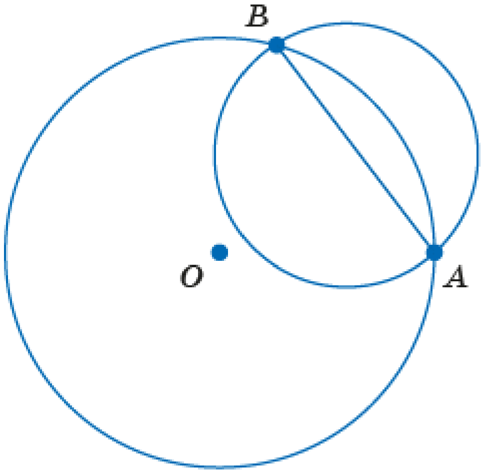


Fig. 7.31

19. The doors of some buses open inward. Figure 7.32 shows the initial closed position of doors AB, CD and the intermediate position $A'B'$ and $C'D'$ of opening doors. Find the envelope of all possible positions of the opening doors.

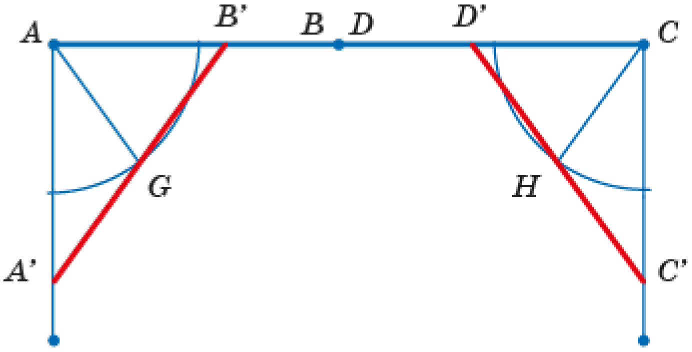


Fig. 7.32

8. Curves of constant width

Let us define the concept of the width of a figure. To do this, consider two parallel lines between which the given figure is located. We will shift them towards each other until they touch the figure. The distance between the resulting parallel lines is the ***width of the figure*** in the direction perpendicular to these lines (Fig. 8.1).

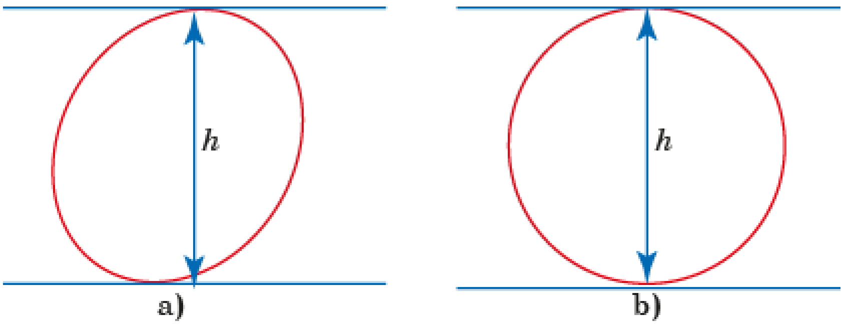


Fig. 8.1

For different directions, the width of a figure can be different. An example of a figure with the same (constant) width in all directions is a circle. Its width equals its diameter.

Are there curves, different from a circle, that have constant width? It turns out that there are. An example of such a curve is the curve devised by the French scientist F. Reuleaux (1829–1905), called the "***Reuleaux triangle***" (Fig. 8.2).

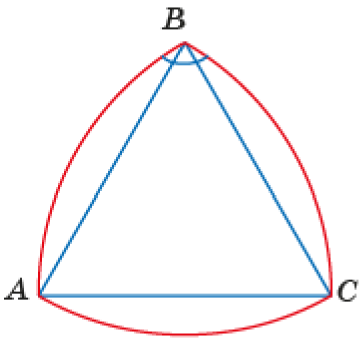


Fig. 15.2

To construct it, consider an equilateral triangle ABC with side a. With center at vertex A and radius a, draw arc BC of a circle. Similarly, with centers at vertices B and C and radius a, draw circular arcs AC and AB. The result is the desired curve, consisting of three circular arcs. Its width equals the side a of the equilateral triangle.

Theorem. The perimeter of the Reuleaux triangle equals the circumference of a circle whose diameter equals the width of the Reuleaux triangle.

Proof. Since the Reuleaux triangle consists of three circular arcs, each of radius a and angle $\frac{\pi}{3}$, their total length equals πa.

Note that the Reuleaux triangle can be inscribed in a square whose side equals the side of this triangle. Moreover, the Reuleaux triangle can rotate inside the square, constantly touching its sides (Fig. 8.3).

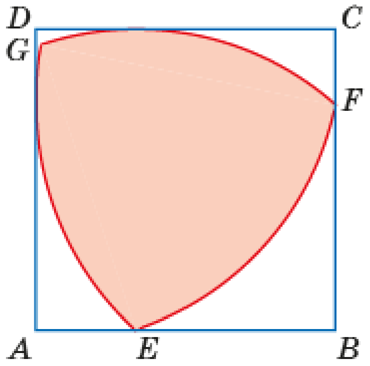


Fig. 8.3

This property of the Reuleaux triangle is used for drilling square holes. You can learn about this and other properties of the Reuleaux triangle on the website https://etudes.ru.

A curve of constant width can also be obtained from an irregular triangle.

Consider three lines intersecting pairwise at points A, B, C. Denote the sides of triangle ABC as a, b, c, respectively (Fig. 8.4).

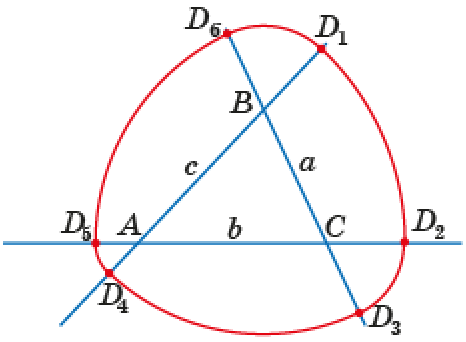


Fig. 8.4

On the extension of segment AB, take point D_1. With center at point A and radius $r = AD_1$, draw a circular arc connecting point D_1 and point D_2 on ray AC.

With center at point C and radius $CD_2 = r - b$, draw a circular arc connecting point D_2 and point D_3 on ray BC.

With center at point B and radius $BD_3 = a + r - b$, draw a circular arc connecting point D_3 and point D_4 on ray BA.

With center at point A and radius $AD_4 = a + r - b - c$, draw a circular arc connecting point D_4 and point D_5 on ray CA.

With center at point C and radius $CD_5 = a + r - c$, draw a circular arc connecting point D_5 and point D_6 on ray CB.

With center at point B and radius $BD_6 = r - c$, draw a circular arc connecting point D_6 and point D_1 on ray AB. We obtain a closed curve (Fig. 8.4).

Curves of constant width can be obtained not only from triangles but also from polygons with an odd number of sides.

Figure 8.5 shows such curves for a regular pentagon and a regular heptagon.

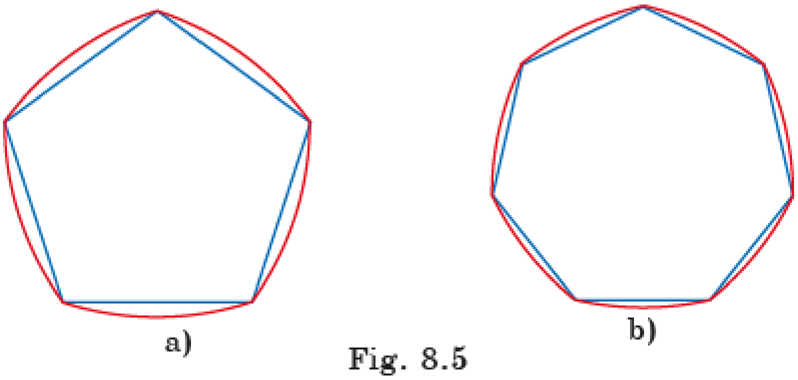

a) b)

Fig. 8.5

The British 20 pence coin is made in the shape of a regular Reuleaux heptagon (Fig. 8.6). The property of constant width allows it to be used in various vending machines.

Fig. 8.6

Let us consider another method of obtaining a curve of constant width.

Take four lines intersecting pairwise at points A, B, C, D. Denote the sides of quadrilateral $ABCD$ as a, b, c, d, respectively (Fig. 8.7).

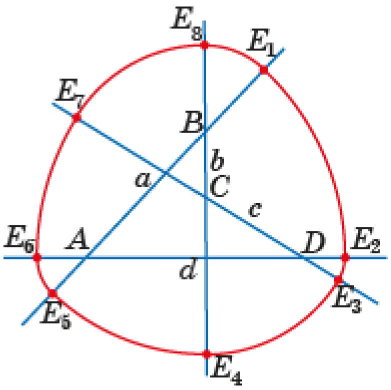


Fig. 8.7

On the extension of segment AB, take point E_1. With center at point A and radius $r = AE_1$, draw a circular arc connecting point E_1 and point E_2 on ray AD.

Perform the further construction of the curve of constant width independently, and find its width. Prove that its length equals the circumference of a circle with diameter equal to the width h of the curve.

The desired curve is shown in Figure 8.7, where E_2E_3 is a circular arc with center D and radius $r - d$, E_3E_4 is a circular arc with center C and radius $c + r - d$, E_4E_5 is a circular arc with center B and radius $b + c + r - d$, E_5E_6 is a circular arc with center A and radius $b + c + r - d - a$, E_6E_7 is a circular arc with center D and radius $b + c + r - a$, E_7E_8 is a circular arc with center C and radius $b + r - a$, E_8E_1 is a circular arc with center B and radius $r - a$. The width h of the curve equals $2r + b + c - a - d$.

Exercises

1. Find the angles of the Reuleaux triangle formed by the tangents to the circular arcs at its vertices (Fig. 8.8).

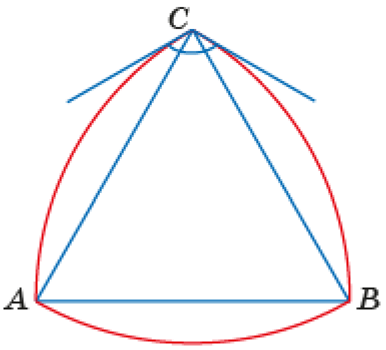


Fig. 8.8

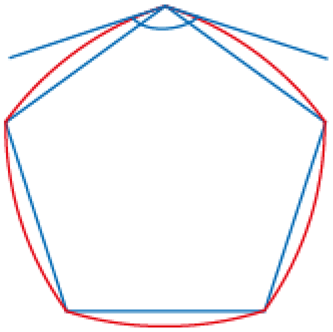


Fig. 8.9

2. Prove that the curve shown in Figure 8.4 has constant width. Find its expression in terms of a, b, c, and r.

3. Prove that the length of the curve shown in Figure 8.4 equals the circumference of a circle with diameter equal to the width h of the curve.

4. The side of a regular pentagon is 1. Find the width of the corresponding Reuleaux pentagon (Fig. 8.5, a).

5. Find the angles of the Reuleaux pentagon formed by the tangents to the circular arcs at its vertices (Fig. 8.9).

6. For the lines shown in Figure 8.10, construct the Reuleaux triangle.

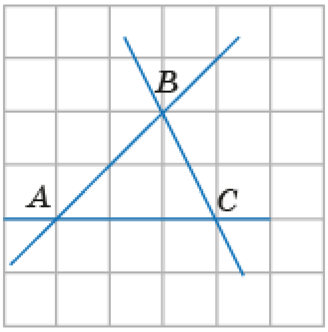


Fig. 8.10

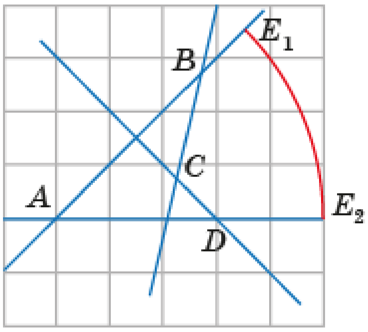


Fig. 8.11

7. For the lines shown in Figure 8.11, continue the construction of the curve of constant width. Find it’s width *h* if $AB = a$, $BC = b$, $CD = c$, $DE = d$, $EA = e$, $BF_1 = r$.

8. For the lines shown in Figure 8.12, continue the construction of the curve of constant width. Find it’s width *h* if $AB = a$, $BC = b$, $CD = c$, $DA = d$, $BE_1 = r$.

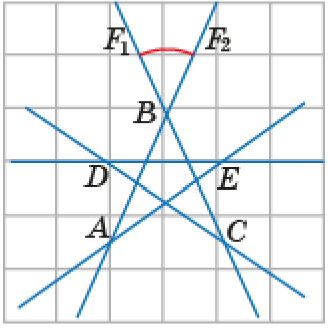


Fig. 8.12

9. The website "Mathematical Etudes" shows that if rollers with a Reuleaux triangle profile are made, a book will roll on them at a constant height equal to the width of the Reuleaux triangle (Fig. 8.13).

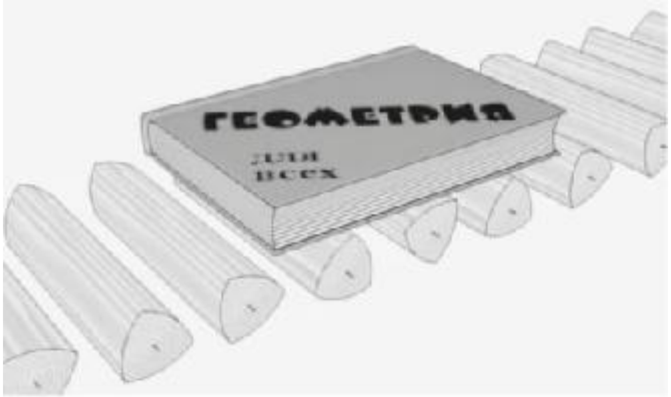


Fig. 8.13

In the GeoGebra computer program, try to model this motion of the Reuleaux triangle along a straight line (Fig. 8.14).

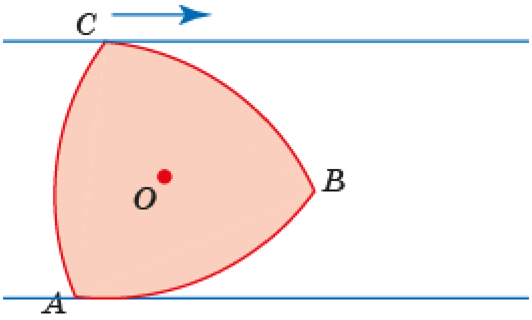


Fig. 8.14

10. In the GeoGebra computer program, try to model the motion of the Reuleaux triangle inside a square so that it touches the sides of this square (Fig. 8.3).

MODULE 3. ANALYTICAL DESCRIPTION OF CURVES

9. Analytical description of a straight line

Let's consider several ways to analytically describe a line on the coordinate plane.

1. Let's derive the equation of a line passing through a given point $A_0(x_0, y_0)$ and perpendicular to a given vector $\vec{n}(a, b)$, which is called the ***normal vector*** of this line (Fig. 9.1).

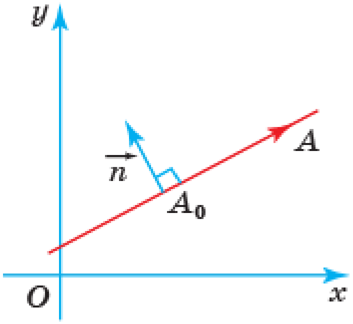


Fig. 9.1

A point $A(x, y)$ belongs to this line if and only if the vector $\overrightarrow{A_0A}(x - x_0, y - y_0)$ is perpendicular to the vector $\vec{n}(a, b)$, i.e., if their dot product equals zero.

Expanding the dot product using the coordinates of these vectors, we obtain the equation $a(x - x_0) + b(y - y_0) = 0$.

Denoting $-ax_0 - by_0 = c$, this equation can be rewritten as

$$ax + by + c = 0.$$

Thus, the following theorem holds.

Theorem. A line on a plane is given by the equation

$ax + by + c = 0$,

where a, b, c are some numbers, and a, b are not both zero and constitute the coordinates of the normal vector $\vec{n}$.

If $b \neq 0$, then by dividing by b, the equation of the line is reduced to the form $y = kx + l$.

The coefficient k is called the ***slope*** of this line.

Let's clarify its geometric meaning.

Consider two points $A_1(x_1, y_1)$, $A_2(x_2, y_2)$ on the line, where $x_1 \neq x_2$ (Fig. 9.2).

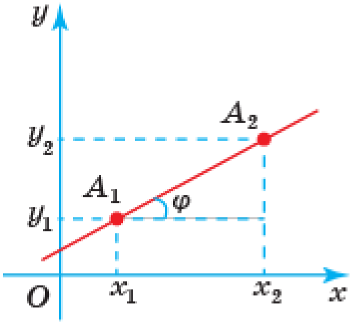


Fig. 9.2

Their coordinates satisfy the line equation, i.e.,

$$y_1 = kx_1 + l,\ y_2 = kx_2 + l.$$

Subtracting the first equality from the second yields $y_2 - y_1 = k(x_2 - x_1)$. Consequently,

$$k = \frac{y_2 - y_1}{x_2 - x_1}.$$

Thus, the slope k equals the tangent of the angle φ that this line makes with the x-axis.

A line passing through a given point with a given normal vector can be obtained in the GeoGebra software. To do this:

- Mark a point using the "Point" tool;
- Construct a vector using the "Vector" tool;
- Select the "Perpendicular Line" tool and left-click on the point and the vector. The desired line will appear on the screen. Its equation will appear on the left side of the screen (Fig. 9.3).

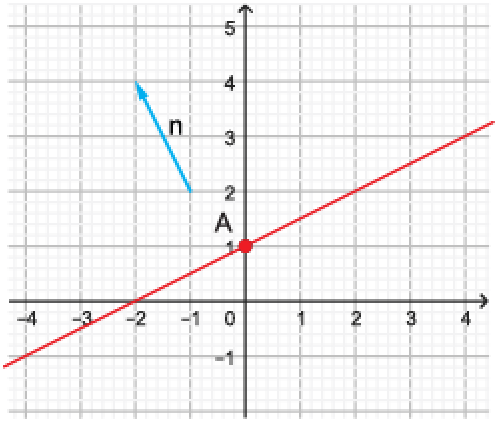


Fig. 9.3

2. Let's derive the equation of a line passing through a given point $A_0(x_0, y_0)$ parallel to a given vector $\vec{m}(c, d)$ or containing it (Fig. 9.4). This vector is called the ***direction vector*** of this line.

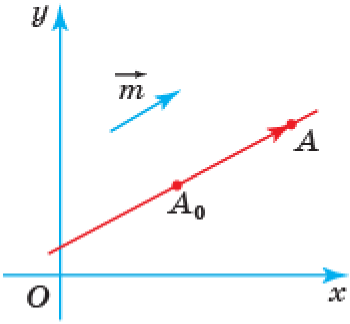


Fig. 9.4

Note that the dot product of vectors $\vec{m}(c, d)$ and $\vec{n}(d, -c)$ equals zero. Therefore, vector $\vec{n}$ is perpendicular to vector $\vec{m}$, meaning it is a normal vector of this line.

The desired equation of the line will be

$$d(x - x_0) - c(y - y_0) = 0.$$

Thus, the following theorem holds.

Theorem. A line passing through point $A_0(x_0, y_0)$ with a given direction vector $\vec{m}(c, d)$ is given by the equation

$$d(x - x_0) - c(y - y_0) = 0.$$

The resulting equation can be written as a second-order determinant equal to zero

$$\begin{vmatrix} x - x_0 & y - y_0 \\ c & d \end{vmatrix} = 0.$$

Theorem. A line passing through a point $A_0(x_0, y_0)$ with a given direction vector $\vec{m}(c, d)$ is given by the equation

$d(x - x_0) - c(y - y_0) = 0.$

The resulting equation can be written as a second-order determinant equal to zero.

$$\begin{vmatrix} x - x_0 & y - y_0 \\ c & d \end{vmatrix} = 0.$$

A line passing through a given point with a given direction vector can be obtained using the computer program GeoGebra. To do this, select the "Parallel Line" tool and, with the left mouse button, specify the point and the vector. As a result, the desired line will appear on the screen. Its equation will appear on the left side of the screen (Fig. 9.5).

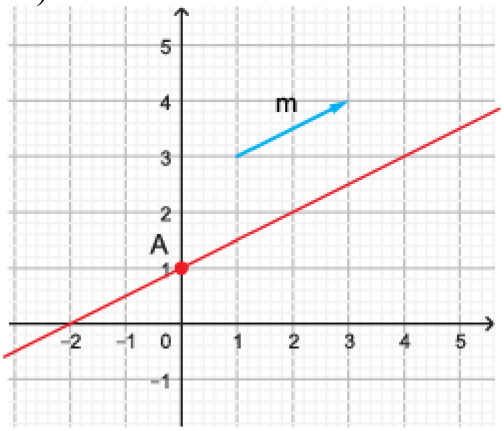


Fig. 9.5

3. Let us derive the equation of a line passing through two points $A_1(x_1, y_1)$, $A_2(x_2, y_2)$ (Fig. 9.6).

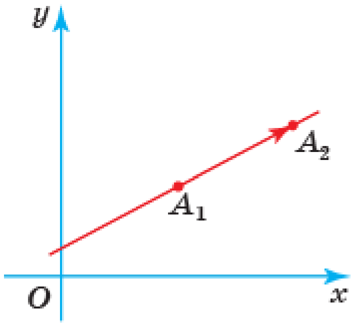


Fig. 9.6

In this case, the vector $\vec{m}(x_2 - x_1, y_2 - y_1)$ can be taken as the direction vector. Let us take point $A_1(x_1, y_1)$ as the point A_0 belonging to this line. Substituting the coordinates of the direction vector and point A_1 into the equation of a line passing through a given point with a given direction vector, we obtain the equation of a line passing through two points:

$$(y_2 - y_1)(x - x_1) - (x_2 - x_1)(y - y_1) = 0.$$

Thus, the following theorem holds.

Theorem. A line passing through points $A_1(x_1, y_1)$, $A_2(x_2, y_2)$ is given by the equation

$$(y_2 - y_1)(x - x_1) - (x_2 - x_1)(y - y_1) = 0.$$

The resulting equation can be written as a second-order determinant equal to zero.

$$\begin{vmatrix} x - x_1 & y - y_1 \\ x_2 - x_1 & y_2 - y_1 \end{vmatrix} = 0.$$

A line passing through two given points can be obtained in the computer program GeoGebra. To do this, select the "Line" tool and, with the left mouse button, specify two points. As a result, the desired line will appear on the screen. Its equation will appear on the left side of the screen (Fig. 9.7).

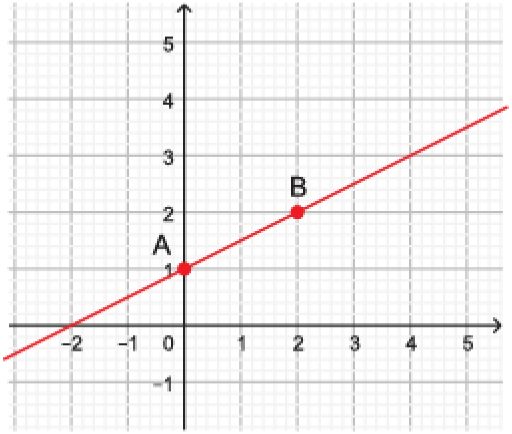


Fig. 9.7

4. Let us present another way of defining a line that intersects the coordinate axes (Fig. 9.8) at points $A(a, 0)$ and $B(0, b)$.

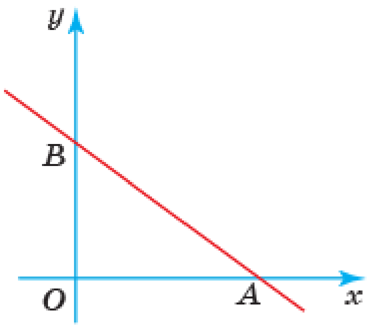


Fig. 9.8

It is directly verified that the required equation of this line is the equation

$$\frac{x}{a}+\frac{y}{b}=1.$$

A line given by the equation $ax + by + c = 0$ can be obtained in the computer program GeoGebra. For example, if you type the equation $x + 2y – 6 = 0$ in the "Input" bar, you will obtain the line shown in Figure 9.9.

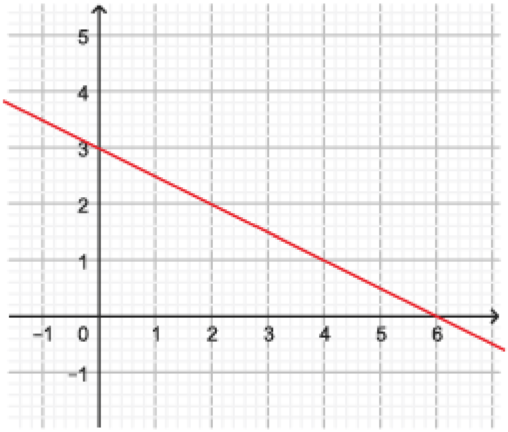


Fig. 9.9

A line can also be obtained for the general equation $ax + by + c = 0$. To do this, you need to create sliders a, b, c and type in the "Input" bar: $ax + by + c = 0$.

As a result, you will get a line defined by this equation. As the slider values change, the line itself will change.

Let us determine the relative position of lines on a plane, depending on their equations.

Two lines on a plane will be parallel if their normal vectors $\overrightarrow{n_1}$, $\overrightarrow{n_2}$ are collinear, i. e., for some number t the equality $\overrightarrow{n_1} = t\overrightarrow{n_2}$ holds.

For lines given by the equations $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, the normal vectors have coordinates (a_1, b_1), (a_2, b_2). Consequently, such lines are parallel if for some number t the equalities hold: $a_2 = ta_1$, $b_2 = tb_1$.

In this case, if $c_2 = tc_1$, then these equations define the same line. If $c_2 \neq tc_1$, then these equations define parallel lines.

If two lines intersect, the angle φ between them can be calculated using the scalar product formula of their normal vectors. Namely,

$$\cos\varphi = \frac{|\overrightarrow{n_1}\cdot\overrightarrow{n_2}|}{|\overrightarrow{n_1}|\cdot|\overrightarrow{n_2}|} = \frac{|a_1a_2 + b_1b_2|}{\sqrt{a_1^2+b_1^2}\cdot\sqrt{a_2^2+b_2^2}}.$$

In particular, the lines will be perpendicular if the scalar product of vectors $\overrightarrow{n_1}, \overrightarrow{n_2}$ is zero, i.e., the equalities hold:
$\overrightarrow{n_1}\cdot\overrightarrow{n_2} = a_1a_2 + b_1b_2 = 0$.

An approximate value of the angle between two intersecting lines can be obtained in the computer program GeoGebra. To do this, select the "Angle" tool and, with the left mouse button, specify two lines. As a result, the desired approximate value of the angle between the two given lines will appear on the screen (Fig. 9.10).

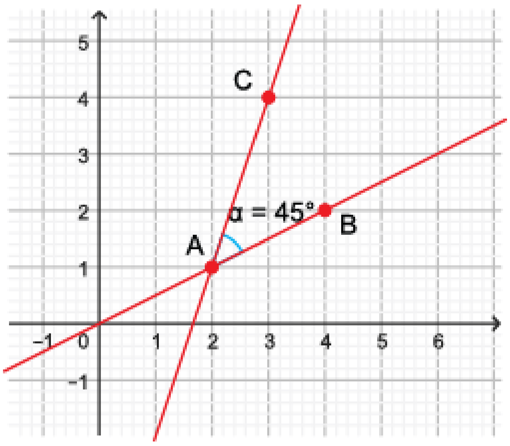


Fig. 9.10

Let us derive the formula for the distance h from a given point $A_1(x_1, y_1)$ to a given line l, given by the equation $ax + by + c = 0$ (Fig. 9.11).

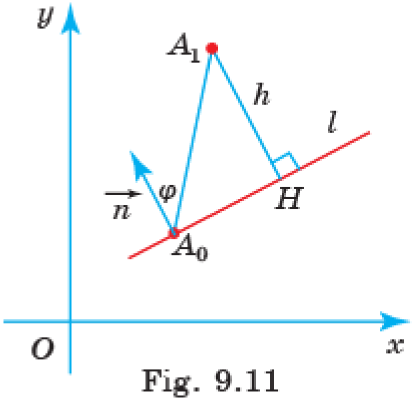


Fig. 9.11

Let $A_0(x_0, y_0)$ be some point on the line, $\vec{n}(a,\ b)$ be the normal vector, and φ be the angle between the line A_0A_1 and the normal vector. From point A_1, drop a perpendicular A_1H to the given line. The desired distance h equals $A_1H = \left|\overrightarrow{A_0A_1}\right| \cdot \cos\varphi$. From the scalar product formula we find

$$\cos\varphi = \frac{|\vec{n} \cdot \overrightarrow{A_0A_1}|}{|\vec{n}| \cdot |\overrightarrow{A_0A_1}|} = \frac{|a(x_1 - x_0) + b(y_1 - y_0)|}{\sqrt{a^2 + b^2} \cdot |\overrightarrow{A_0A_1}|}.$$

Considering that $-ax_0 - by_0 = c$, we obtain the desired formula

$$h = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.$$

An approximate value of the distance from a point to a line can be obtained in the GeoGebra program. To do this, select the "Distance or Length" tool and, with the left mouse button, specify the point and the line. As a result, the desired approximate value of the distance from the given point to the given line will appear on the screen (Fig. 9.12).

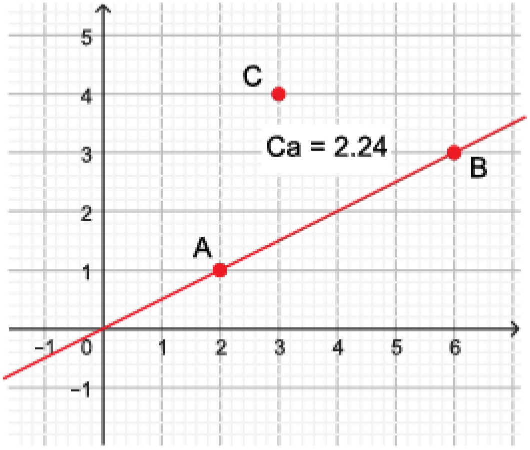


Fig. 9.12

Exercises

1. What equations do the coordinate lines have: a) *Ox*; b) *Oy*?

2. Write the equation of the line: a) passing through point $A(a, 0)$ and parallel to the *Oy* axis; b) passing through point $B(0, b)$ and parallel to the *Ox* axis.

3. Write the equation of the line passing through point $A_0(2, 1)$ with normal vector $\vec{n}(1, -1)$. Obtain it in the computer program GeoGebra.

4. Write the equation of the line passing through point $A_0(2, 1)$ and perpendicular to the line given by the equation: a) $x + y + 1 = 0$; b) $2x - 3y + 4 = 0$. Obtain it in the computer program GeoGebra.

5. Write the equation of the line passing through point $A(-2, 1)$, which is the foot of the perpendicular dropped from the origin to this line.

6. Point $H(-2, 4)$ is the foot of the perpendicular dropped from the origin to the line. Write the equation of this line.

7. Write the equation of the line passing through point $C(-1, 3)$ and parallel to: a) the *Ox* axis; b) the *Oy* axis; c) the line $y = 2x$.

8. Write the equation of the line passing through point $A_0(2, -1)$ with direction vector $\vec{m}(-1, 2)$. Obtain it in the computer program GeoGebra.

9. Write the equation of the line passing through point $A_0(-1, 2)$ and parallel to the line given by the equation: a) $x + y + 1 = 0$; b) $2x - 3y + 4 = 0$. Obtain it in the computer program GeoGebra.

10. Write the equation of the line passing through the origin with slope: a) $k = 1$; b) $k = 2$; c) $k = 0.5$; d) $k = -1$; e) $k = -2$; f) $k = -0.5$. Draw these lines on graph paper.

11. Find the slope of the line: a) $3x - 2y + 4 = 0$; b) $2x + y - 1 = 0$.

12. Write the equation of the line passing through points $A(0, 1)$, $B(1, 0)$.

13. Write the equation of the line passing through points $M(-1, 3)$, $N(1, 4)$. Obtain it in the computer program GeoGebra. Find the coordinates of the normal vector of this line.

14. Write the equations of lines a_1, a_2 shown in Figure 9.13.

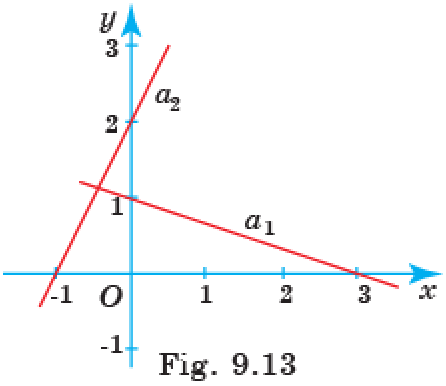


Fig. 9.13

15. Determine which of the following pairs of lines: a) are parallel; b) are perpendicular:

1) $x + y - 2 = 0$, $x + y + 3 = 0$;
2) $x + y - 2 = 0$, $x - y - 3 = 0$;
3) $-7x + y = 0$, $7x - y + 4 = 0$;
4) $4x - 2y - 8 = 0$, $-x - 2y + 4 = 0$.

16. Find the coordinates of the intersection point of the lines: a) $x - y - 1 = 0$, $x + y + 3 = 0$;
b) $x - 3y + 2 = 0$, $2x - 5y + 1 = 0$.

17. Write the equation of the perpendicular bisector of segment AB: a) $A(1, 2)$, $B(4, 3)$; b) $A(3, 2)$, $B(-1, 4)$.

18. The vertices A, B, C of triangle ABC have coordinates (0, 2), (4, 2), (3, 6) respectively. Find: a) the equations of the lines containing the medians of this triangle, the coordinates of their intersection point M; b) the equations of the lines containing the altitudes of this triangle, the coordinates of their intersection point H; c) the equations of the perpendicular bisectors of the sides of this triangle; d) the coordinates of the center of the circle circumscribed about this triangle.

19. Find the angle between the lines given by the equations: a) $2x + y - 1 = 0$, $x - 2y + 3 = 0$; b) $x + y + 1 = 0$, $x - y - 1 = 0$. Draw these lines.

20. Find the cosine of the angle between the lines given by the equations: a) $x + y + 1 = 0$, $3x - 4y + 5 = 0$; b) $x + 2y - 1 = 0$, $x - 3y - 5 = 0$. Obtain the approximate value of the angle in the computer program GeoGebra.

21. Find the distance from the point: a) $O(0, 0)$; b) $A(3, -1)$ to the line given by the equation $3x + 4y - 12 = 0$. Obtain the approximate value of the distance in the computer program GeoGebra.

22. Write the equation of the line symmetric to the line given by the equation $ax + by + c = 0$ with respect to: a) the Ox axis; b) the Oy axis; c) the origin O.

10. Curves Defined by Equations in Cartesian Coordinates

Here we will examine several classical curves defined by equations in Cartesian coordinates.

1. Circle. A circle with center $P(x_0, y_0)$ and radius R (Fig. 10.1) is given by the equation $(x - x_0)^2 + (y - y_0)^2 = R^2$.

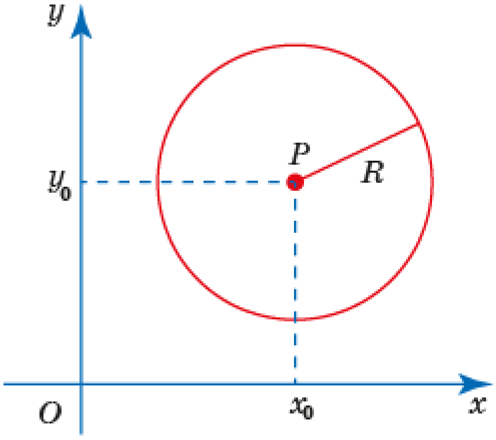


Fig. 10.1

2. Parabola. Recall that a parabola is the locus of points equidistant from a given line d and a given point F not lying on d. The line d is called the directrix, and the point F is the focus of the parabola. The line passing through the focus of the parabola and perpendicular to the directrix is called the axis of the parabola.

Let us derive the equation of a parabola. Denote by G the intersection point of the parabola's axis with its directrix. Let the length of segment FG be $2a$. Introduce a coordinate system with the origin at the midpoint O of segment FG, the x-axis as the line parallel to the directrix and passing through the origin, and the y-axis as the axis of the parabola (Fig. 10.2). Then the focus F will have coordinates $(0, a)$.

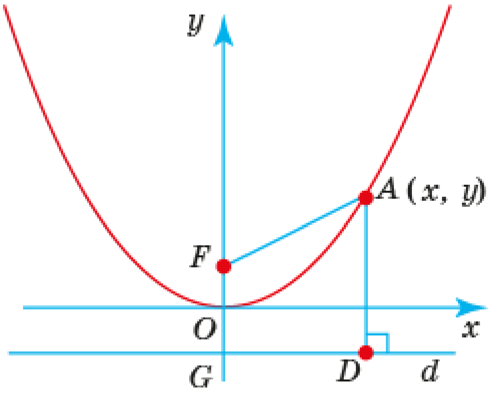


Fig. 10.2

Let $A(x, y)$ be a point in the plane. Its distances to the focus and the directrix are $\sqrt{x^2 + (y - a)^2}$ and $|y+a|$, respectively. Point A belongs to the parabola if and only if the equality $\sqrt{x^2 + (y - a)^2} = |y+a|$ holds. Squaring both sides of this equality and combining like terms yields the equality $4ay = x^2$, which is the desired equation of the parabola.

To obtain the parabola defined by this equation in the GeoGebra software, create a slider a. In the "Input" bar, type 4ay=x^2 and press "Enter". The parabola will appear on the screen (Fig. 10.3). The value of slider a can be changed. The shape of the parabola will change accordingly.

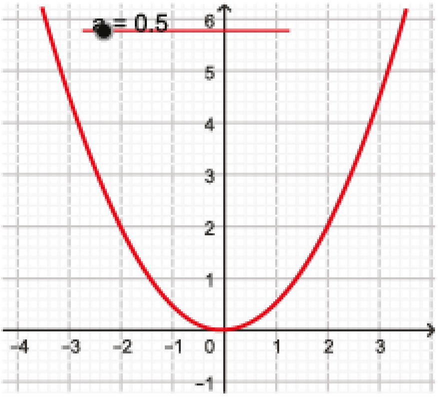


Fig. 10.3

3. Ellipse. Recall that an ellipse is the locus of points in the plane for which the sum of the distances to two fixed points F_1, F_2 equals a given constant greater than the distance between the points F_1, F_2. The points F_1, F_2 are called the foci of the ellipse. The segment connecting the points of the ellipse and passing through its foci is called the major axis of the ellipse. The segment connecting the points of the ellipse and perpendicular to the major axis is called the minor axis of the ellipse.

Let us derive the equation of an ellipse. Let $F_1(-c, 0)$, $F_2(c, 0)$ be the foci of the ellipse. Take the given constant of the ellipse to be $2a$ (Fig. 10.4).

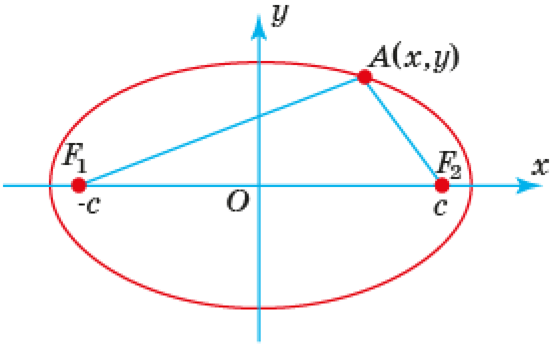


Fig. 10.4

Let $A(x, y)$ be a point in the plane. Its distances to the foci are respectively $\sqrt{(x-c)^2+y^2}$ and $\sqrt{(x+c)^2+y^2}$. Point A belongs to the ellipse if and only if the equality holds

$$\sqrt{(x-c)^2+y^2}+\sqrt{(x+c)^2+y^2}=2a.$$

Move the second term on the left side of this equality to the right side and square both sides of the resulting equality. We obtain the equality

$$(x-c)^2+y^2=4a^2-4a\sqrt{(x+c)^2+y^2}+(x+c)^2+y^2.$$

Combine like terms to obtain the equality

$$a\sqrt{(x+c)^2+y^2}=a^2+xc.$$

Square again and combine like terms. We obtain the equality

$$x^2(a^2 - c^2) + y^2a^2 = a^4 - a^2c^2.$$

Denote $b^2 = a^2 - c^2$ and divide both sides of this equality by a^2b^2. We obtain the equality

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

which is the desired equation of the ellipse.

To obtain the ellipse defined by this equation in the GeoGebra software, create sliders *a* and *b*. In the "Input" bar, type x^2/a^2+y^2/b^2=1 and press "Enter". The ellipse will appear on the screen (Fig. 10.5). The values of *a* and *b* can be changed. The shape of the ellipse will also change accordingly.

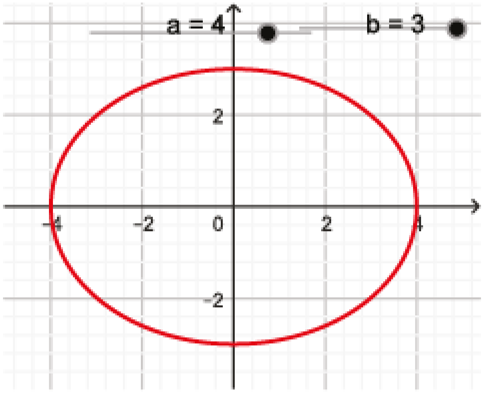


Fig. 10.5

4. Hyperbola. Recall that a hyperbola is the locus of points in the plane for which the absolute value of the difference of the distances to two fixed points F_1, F_2 equals a given positive constant less than the distance between the points F_1, F_2. The points F_1, F_2 are called the foci of the hyperbola.

Let us derive the equation of a hyperbola. Let $F_1(-c, 0)$, $F_2(c, 0)$ be the foci of the hyperbola. Take the given positive constant of the hyperbola to be $2a$ (Fig. 10.6).

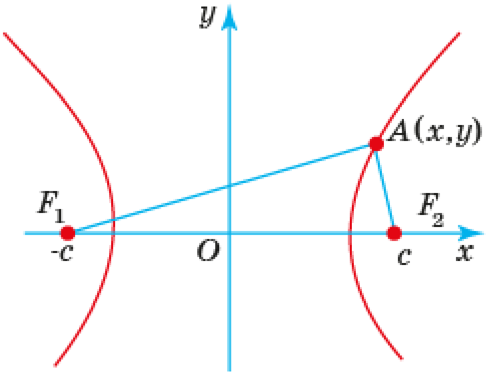


Fig. 10.6

Point $A(x, y)$ belongs to the hyperbola if and only if the equality holds

$$\left|\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2}\right| = 2a.$$

Rewrite it as

$$\sqrt{(x+c)^2 + y^2} = \pm 2a + \sqrt{(x-c)^2 + y^2}$$

and square both sides of this equality. We obtain

$$(x+c)^2 + y^2 = 4a^2 \pm 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2.$$

Expanding the brackets and combining like terms, we obtain the equality

$$xc - a^2 = \pm a\sqrt{(x-c)^2 + y^2}.$$

Squaring again and denoting $b^2 = c^2 - a^2$, we obtain

$$x^2b^2 - y^2a^2 = a^2b^2.$$

Dividing both sides by a^2b^2, we finally obtain the equation of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

To obtain the hyperbola defined by this equation in the GeoGebra software, create sliders *a* and *b*. In the "Input" bar, type x^2/a^2-y^2/b^2=1 and press "Enter". The hyperbola will appear on the screen (Fig. 10.7).

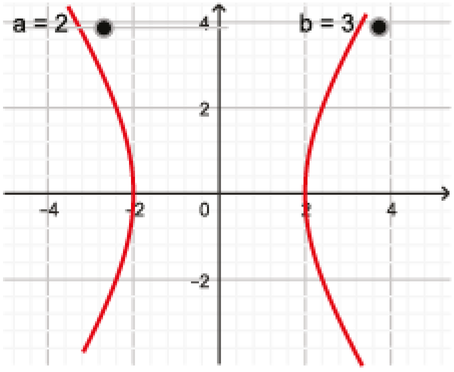


Fig. 10.7

The values of *a* and *b* can be changed. The shape of the hyperbola will also change accordingly.

Let us determine what curves can be defined by the equation

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{10}x + 2a_{20}y + a_{00} = 0 \ (1),$$

where the numbers $a_{11},\ a_{12},\ a_{22}$ are not all zero simultaneously.

We will prove that by rotating the curve defined by this equation about the origin through some angle φ, we can eliminate the term $2a_{12}xy$, i.e., obtain a curve defined by the equation

$$a'_{11}x'^2 + a'_{22}y'^2 + 2a'_{10}x' + 2a'_{20}y' + a_{00} = 0 \ (2),$$

where the numbers $a'_{11},\ a'_{22}$ are not both zero.

Let point *A′* be obtained by rotating point *A*(*x*, *y*) about the origin through an angle φ (Fig. 10.8).

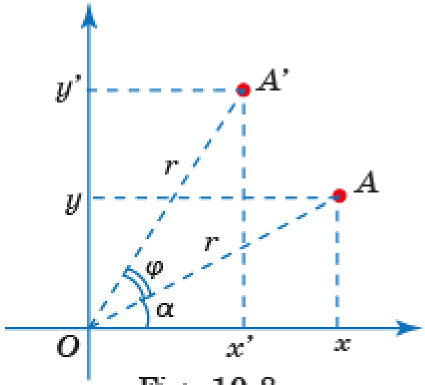


Fig. 10.8

Let us find the coordinates (x', y') of point A'. We have

$$x' = r \cdot \cos(\alpha + \varphi) = r \cdot (\cos\alpha \cdot \cos\varphi - \sin\alpha \cdot \sin\varphi) = x \cdot \cos\ \varphi - y \cdot \sin\varphi,$$
$$y' = r \cdot \sin(\alpha + \varphi) = r \cdot (\sin\alpha \cdot \cos\varphi + \cos\alpha \cdot \sin\varphi) = x \cdot \sin\ \varphi + y \cdot \cos\varphi.$$

Let us express the coordinates (x, y) from these equalities in terms of the coordinates (x', y'). We get

$$x = x' \cdot \cos\varphi + y' \cdot \sin\varphi,$$
$$y = -x' \cdot \sin\varphi + y' \cdot \cos\varphi.$$

Substitute these expressions into equation (1). We obtain the equation

$$a'_{11}x'^2 + a'_{12}x'y' + a'_{22}y'^2 + 2a'_{10}x' + 2a'_{20}y' + a_{00} = 0,$$

where

$$a'_{11} = a_{11}\cos^2\varphi - 2a_{12}\cos\varphi \cdot \sin\varphi + a_{22}\sin^2\varphi,$$
$$a'_{12} = 2a_{11}\cos\varphi \cdot \sin\varphi - 2a_{22}\sin\varphi \cdot \cos\varphi + 2a_{12}(\cos^2\varphi - \sin^2\varphi),$$
$$a'_{22} = a_{11}\sin^2\varphi + 2a_{12}\cos\varphi \cdot \sin\varphi + a_{22}\cos^2\varphi,$$
$$a'_{10} = 2a_{10}\cos\varphi - 2a_{20}\sin\varphi,\ a'_{20} = 2a_{10}\sin\varphi + 2a_{20}\cos\varphi.$$

Let us rewrite the expression for the coefficient a'_{12}in the form

$$a'_{12} = \sin 2\varphi(a_{11} - a_{22}) + 2a_{12} \cdot \cos 2\varphi.$$

If the coefficient $a_{12}$is non-zero, then by choosing φ such that

$$\operatorname{ctg} 2\varphi = \frac{a_{22} - a_{11}}{2a_{12}},$$

we obtain $a'_{12} = 0$.

Expressions for $\cos\varphi$ and $\sin\varphi$ can be obtained using trigonometric formulas: $\operatorname{tg}^2\varphi = \frac{1}{\operatorname{ctg}^2\varphi}$, $\cos^2 2\varphi = \frac{1}{1 + \operatorname{tg}^2 2\varphi}$, $\cos^2\varphi = \frac{1 + \cos 2\varphi}{2}$, $\sin^2\varphi = \frac{1 - \cos 2\varphi}{2}$.

Let us prove that if $a'_{11} \neq 0$, then by a parallel translation of the curve defined by equation (2) we can eliminate the term $2a'_{10}x'$, i. e., obtain a curve defined by the equation

$$a'_{11}x''^2 + a'_{22}y'^2 + 2a'_{20}y' + a''_{00} = 0 \ (3).$$

Rewrite equation (2) in the form

$$a'_{11}(x' + \frac{a'_{10}}{a'_{11}})^2 + a'_{22}y'^2 + 2a'_{20}y' + a''_{00} = 0,$$

where $a''_{00} = a_{00} - \frac{{a'_{10}}^2}{a'_{11}}$.

Denote $x'' = x' + \frac{a'_{10}}{a'_{11}}$. We obtain the desired equation (3). The corresponding curve is obtained from the curve defined by equation (3) by a parallel translation by the vector $\vec{m}(\frac{a'_{10}}{a'_{11}}, 0)$.

Similarly, if $a'_{22} \neq 0$, then by a parallel translation of the curve defined by equation (2) we can eliminate the term $2a'_{20}y'$, i.e., obtain a curve defined by the equation

$$a'_{11}x'^2 + a'_{22}y''^2 + 2a'_{10}x' + a''_{00} = 0 \ (4).$$

If both coefficients a'_{11} and a'_{22} are non-zero, then equation (2) is reduced to the form

$$a'_{11}x''^2 + a'_{22}y''^2 + a''_{00} = 0 \ (5).$$

Depending on the coefficients a'_{11} a'_{11}, a'_{22}, a''_{00} it can define a circle, an ellipse, a hyperbola, two straight lines passing through the origin, a point $O(0, 0)$, or no curve at all.

If $a'_{11} \neq 0$, $a'_{22} = 0$, then equation (3) will take the form

$$a'_{11}x''^2 + 2a'_{20}y' + a''_{00} = 0 \ (6).$$

Depending on the coefficients it can define a parabola, a straight line, two parallel lines, or no curve at all.

If $a'_{22} \neq 0$, $a'_{11} = 0$, then equation (4) will take the form

$$a'_{22}y''^2 + 2a'_{10}x' + a''_{00} = 0 \ (7).$$

Depending on the coefficients it can define a parabola, a straight line, two parallel lines, or no curve at all.

Example. Determine which curve is defined by the equation

$$73x^2 - 72xy + 52y^2 + 100x - 200y + 100 = 0.$$

Solution. Let us eliminate the term $72xy$. To do this, rotate the curve by an angle φ, for which ctg 2φ $= \frac{7}{24}$.Then sin φ $= \frac{4}{5}$, cos φ $= \frac{3}{5}$. The formulas for the rotation of coordinate axes are

$$\begin{cases} x = \frac{3}{5}x' - \frac{4}{5}y', \\ y = \frac{4}{5}x' + \frac{3}{5}y'. \end{cases}$$

Substituting these expressions for *x* and *y* into the given equation, we obtain the equation

$$x'^2 + 4y'^2 - 4x' - 8y' + 4 = 0.$$

Let us complete the squares in the left-hand side of this equation. We get the equation

$$(x' - 2)^2 + 4(y' - 1)^2 - 4 = 0.$$

A parallel translation by the vector with coordinates (-2, -1) reduces the equation to the form

$$x''^2 + 4y''^2 - 4 = 0,$$

where

$$\begin{cases} x'' = x' - 2, \\ y'' = y' - 1. \end{cases}$$

Therefore, the original equation defines an ellipse.

Exercises

1. Prove that the curve defined by the equation $x^2 + y^2 + ax + by + c = 0$, where, $c < \frac{a^2}{4} + \frac{b^2}{4}$, is a circle. Find its center and radius.

2. For the parabola defined by the equation $y = x^2$, find the coordinates of the focus and the equation of the directrix.

3. Determine the type of curve defined by the equation $y^2 = x$. Obtain it in the GeoGebra program.

4. Prove that the curve defined by the equation $y = ax^2 + bx + c$, $a \neq 0$, is a parabola. Find the coordinates of the focus and the equation of the directrix.

5. Prove that the curve defined by the equation $2x^2 + y^2 = 1$ is an ellipse. Find the coordinates of the foci and the constant c. Obtain this curve in the GeoGebra program.

6. Prove that the curve defined by the equation $ax^2 + bx + cy^2 + dy + e = 0$, where a and c are positive and distinct, and $e < \frac{b^2}{4a} + \frac{d^2}{4c}$, is an ellipse.

7. Prove that the locus of points $C(x, y)$ such that the sum of the squares of the distances from C to two fixed points $A_1(-a, 0)$ and $A_2(a, 0)$ is constant and equal to c^2 (where $\frac{c^2}{2} > a^2$) is a circle. Find its center and radius.

8. Determine which curve is defined by the equation $y^2 – x^2 = 1$. Obtain it in the GeoGebra program.

9. Prove that the curve defined by the equation $ax^2 + bx - cy^2 - dy + e = 0$, where a and c are positive and distinct, and $e \neq \frac{b^2}{4a} - \frac{d^2}{4c}$, is a hyperbola.

10. Determine which curve is defined by the equation $xy = 1$. Obtain it in the GeoGebra program.

11. Which curve is defined by the equation $xy – 3x + y – 12 = 0$? Obtain this curve using the GeoGebra program.

12. Which curve is defined by the equation $x^2 + xy + y^2 – 1 = 0$? Obtain this curve using the GeoGebra program.

13. For two given points $A(0, 0)$, $B(b, 0)$ and a positive number $k \neq 1$, find the equation of the curve consisting of points C such that $AC = kBC$. Determine its type.

14. For a line a defined by the equation $y = 0$, a point $B(0, b)$, and a number $k > 1$, find the equation of the curve consisting of all points $C(x, y)$ for which the distance to line a equals the distance to point B multiplied by k. Determine its type. Obtain it in the GeoGebra program.

15. For a line a defined by the equation $y = 0$, a point $B(0, b)$, and a positive number $k < 1$, find the equation of the curve consisting of all points $C(x, y)$ for which the distance to line a equals the distance to point B multiplied by k. Determine its type. Obtain it in the GeoGebra program.

16. Given a point $A(0, \mathrm{a})$ and a line defined by the equation $y = 0$. Find the locus of points $C(x, y)$ for which the sum of the distances to the given point and the given line is constant and equals c.

17. Given points $A(a, 0)$ and $B(b, 0)$. Find the locus of points $C(x, y)$ for which the sum of the squares of the distances to the given points is constant and equals c.

18. Given points $A(a, 0)$ and $B(b, 0)$. Find the locus of points $C(x, y)$ for which the difference of the squares of the distances to the given points is constant and equals c.

19. In the GeoGebra software, obtain the ***folium of Descartes*** – the curve whose points' coordinates satisfy the equation $x^3 + y^3 – 3axy = 0$.

20. Recall that the lemniscate of Bernoulli is the locus of points A in the plane for which the product of the distances AF_1 and AF_2 to given points F_1, F_2 equals the square of half the distance between these points. The points F_1, F_2 themselves

are called the foci. Find the equation of the lemniscate of Bernoulli with foci $F_1(-a, 0)$, $F_2(a, 0)$ (Fig. 10.9). Obtain it in the GeoGebra program.

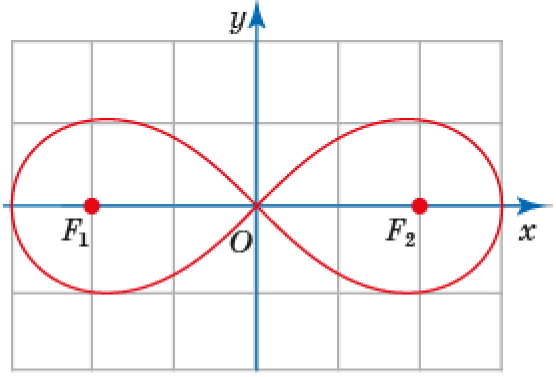


Fig. 10.9

21. On graph paper, draw the curve defined by the equation: a) $|x| + |y| = 1$; b) $2|x| + 3|y| = 6$. Obtain it in the GeoGebra software.

22. On graph paper, draw the curve defined by the equation: a) $|x + y| + |x - y| = 2$; b) $|2x + 3y| + |2x - 3y| = 6$. Obtain it in the GeoGebra software.

23. On graph paper, draw the figure defined by the equation $|x - 1| + |x - 2| + |y - 3| + |y - 4| = 2$. Obtain it in the GeoGebra software.

24. On graph paper, draw the curve defined by the equation $\sqrt{|x|} + \sqrt{|y|} = 1$. Obtain it in the GeoGebra software.

25. On graph paper, draw the curve defined by the equation $x^4 + y^4 = 1$. Obtain it in the GeoGebra software.

11. Curves defined by parametric equations

Let us consider a method of defining a curve as the trajectory of a moving point on a plane. Since the position of a point on the plane is uniquely determined by its coordinates, to define the motion of a point it is sufficient to define the dependences of its coordinates x, y on time t, i.e., to define the functions

$$\begin{cases} x = x(t), \\ y = y(t). \end{cases}$$

In this case, for each value of t we can find the position of the point on the plane.

The argument t is called a parameter. The curve on the plane described by a point whose coordinates satisfy these equations as the parameter *t* varies is called a ***parametrically defined curve*** on the plane (Fig. 11.1). The equations themselves are called ***parametric equations***.

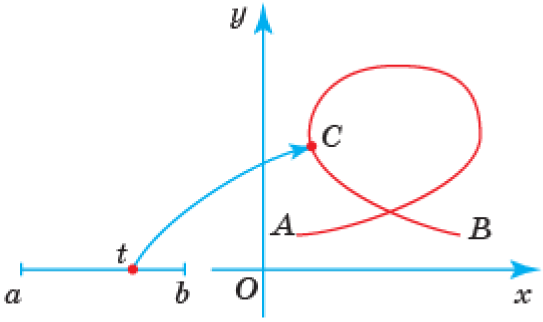


Fig. 11.1

Note that the graph of a function $y = f(x)$ is a special case of a parametrically defined curve on the plane (Fig. 11.2).

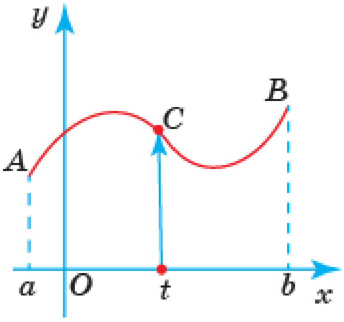


Fig. 11.2

In this case, the parametric equations are

$$\begin{cases} x = t, \\ y = f(t). \end{cases}$$

To obtain a curve defined by parametric equations in the GeoGebra computer program, you need to type in the "Input" bar: Curve(x(t), y(t), t, a, b) and press "Enter". As a result, the desired curve will appear on the screen.

Let us consider examples of defining curves by parametric equations.

1. Line. Consider a line passing through a given point $A_0(x_0, y_0)$, with a given direction vector $\vec{m}(c, d)$ (Fig. 11.3).

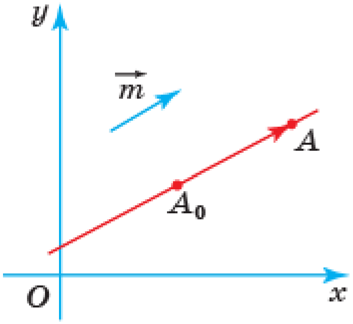


Fig.11.3

A point $A(x, y)$ belongs to this line if and only if the vector $\overrightarrow{A_0A}$ is collinear with the vector $\vec{m}$, i. e., for some number t the equality $\overrightarrow{A_0A} = \vec{m}t$ holds. In coordinate form, this equality can be rewritten as

$$\begin{cases} x - x_0 = ct, \\ y - y_0 = dt. \end{cases}$$

Consequently, the parametric equations of the line are the following equations

$$\begin{cases} x = x_0 + ct, \\ y = y_0 + dt. \end{cases}$$

Note that the parametric equations define not only the line but also the way a point moves along this line. Let $\vec{s}(t)$ denote the vector $\overrightarrow{A_0A}$.

Let us find the expression for the velocity vector of this motion. During the time from t_1 to t_2, the point travels from point $A_1(x_0+ct_1, y_0+dt_1)$ to point $A_2(x_0+ct_2, y_0+dt_2)$. The displacement vector $\overrightarrow{\Delta s}$ will be equal to the vector $\overrightarrow{A_1A_2}$. It has coordinates $(c(t_2 - t_1), d(t_2 - t_1))$. Dividing it by the time increment $\Delta t = t_2 - t_1$, we obtain the velocity vector $\vec{v} = \frac{\overrightarrow{\Delta s}}{\Delta t}$. It has coordinates (c, d) and is equal to the vector $\vec{m}$. Thus, from a physical point of view, the direction vector $\vec{m}(c, d)$ is the velocity vector of the point moving along the line. Its length v is expressed by the formula

$$v = |\vec{v}| = \sqrt{c^2 + d^2}.$$

2. Circle. A circle of radius R centered at the origin (Fig. 11.4) can be defined by the parametric equations

$$\begin{cases} x = R\cos t, \\ y = R\sin t. \end{cases}$$

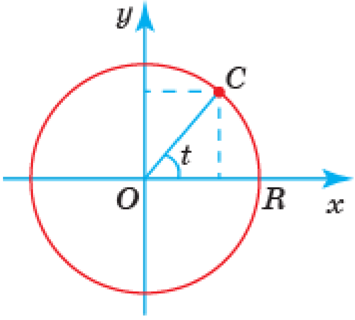


Fig. 11.4

As in the case of the line, these equations define not only the circle but also the motion of a point along this circle. In this case, as the parameter t varies from zero to 2π, the point on the circle makes one revolution counterclockwise, starting and ending at the point with coordinates (R, 0). With a further increase of the parameter t, the point will repeatedly traverse the circle in the counterclockwise direction.

This motion of a point along a circle can be obtained in the GeoGebra computer program. To do this, create a slider t, varying from 0 to 2Pi, and in the "Input" bar type A=(cos(t), sin(t)). Point A, belonging to the circle centered at the origin with radius 1, will appear on the screen. By enabling animation, you will get the motion of this point along the circle.

3. Ellipse. An ellipse defined by the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (Fig. 11.5) can be defined by the parametric equations $\begin{cases} x = a\cos\, t, \\ y = b\sin\, t. \end{cases}$

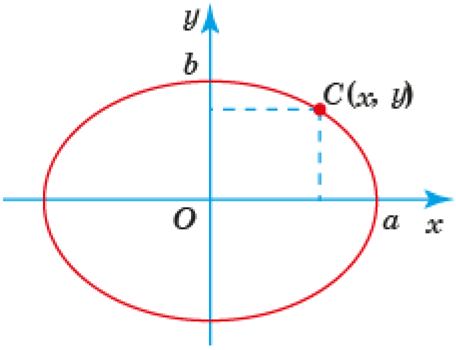


Fig. 11.5

4. Hyperbola. A hyperbola defined by the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ (Fig. 11.6) can be defined by the parametric equations $\begin{cases} x = a\cosh\, t, \\ y = b\sinh\, t, \end{cases}$ where $\mathrm{ch}\, t=\frac{e^t+e^{-t}}{2}$ is the hyperbolic cosine, $sh\, t=\frac{e^t-e^{-t}}{2}$ is the hyperbolic sine.

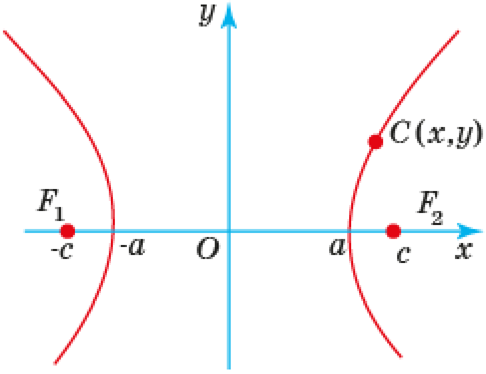


Fig. 11.6

This follows from the fact that for the hyperbolic cosine and sine, the identity $\mathrm{ch}^2 t-\mathrm{sh}^2 t=1$holds, which can be verified directly.

5. Folium of Descartes – a curve defined by the equation $x^3+y^3-3axy=0$ (Fig. 11.7).

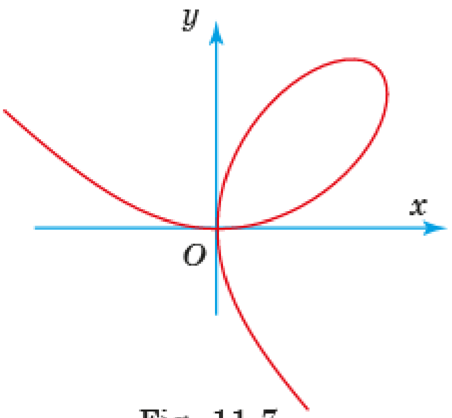


Fig. 11.7

Setting $y = tx$ in this equation and solving for x, we obtain the parametric equations of the folium of Descartes

$$\begin{cases} x = \dfrac{3at}{1+t^3}, \\ y = \dfrac{3at^2}{1+t^3}. \end{cases}$$

6. Cycloid – a curve described by a point fixed on a circle rolling along the Ox axis (Fig. 11.8).

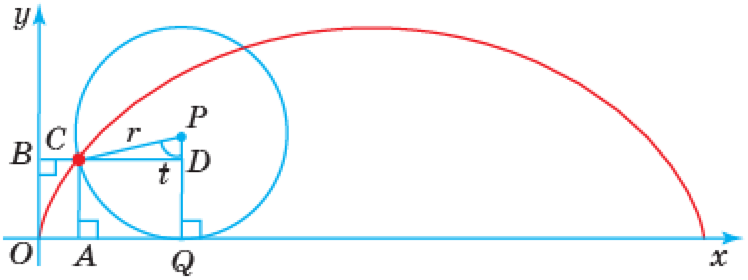


Fig. 11.8

Let the unit circle at the initial moment touch the origin O. Suppose the circle rolls along the Ox axis and moves to point Q. The center of the circle moves to point P. The fixed point on the circle moves to point C.

Let $\angle CPQ = t$. Since the arc $\breve{CQ}$ of the circle has rolled along the segment OQ, their lengths are equal to t. For the coordinates x, y of point C we have

Thus, the parametric equations of the cycloid are

$$\begin{cases} x = t - \sin\ t, \\ y = 1 - \cos\ t. \end{cases}$$

To obtain such a cycloid in the GeoGebra computer program, type in the "Input" bar: Curve(t-sin(t), 1-cos(t), t, 0, 2Pi) and press "Enter". As a result, the cycloid on the interval [0, 2π] will appear on the screen.

In the GeoGebra computer program, you can obtain the trajectory of a point along a cycloid in motion. To do this, you need to:

1) create a slider a, varying from 0 to 2Pi;

2) construct a circle with center (a, 1) and radius 1;

3) mark a point on it with coordinates (a-sin(a), 1-cos(a)) and connect it with a segment to the center of the circle;

4) type in the "Input" bar: Curve(t-sin(t), 1-cos(t), t, 0, a) and press "Enter";

5) enable animation.

As a result, the circle will roll along the line, and the marked point will trace the cycloid.

As a result, the circle will roll along the x-axis, and point C will trace the cycloid (Fig. 11.9).

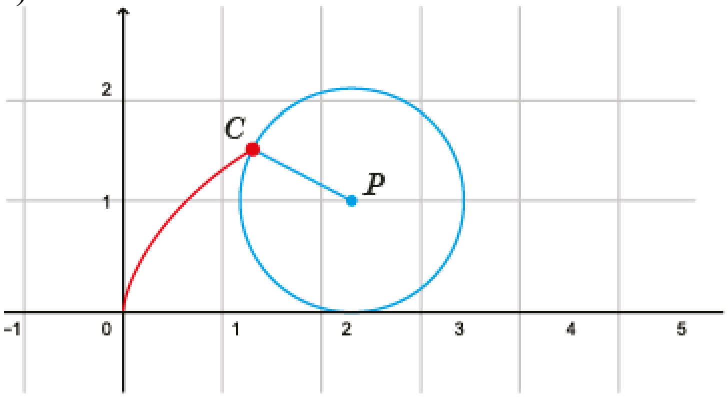


Fig. 11.9

7. Epicycloid – the trajectory of a point fixed on a circle rolling externally along another circle. Let us derive the equation of the epicycloid. Consider a unit circle centered at the origin. Let the rolling circle have radius r. Its center at the initial moment is located at the point with coordinates $(1+r, 0)$, and the fixed point A on it is located at the point with coordinates $(1, 0)$. Suppose the rolling circle rotates through a certain angle t. During this, the center of the rolling circle moves to point P, and the fixed point moves to point $C(x, y)$ (Fig. 11.10).

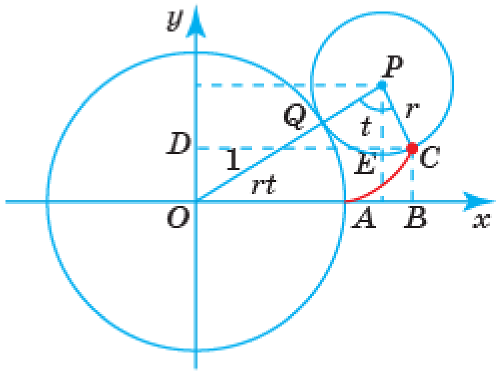


Fig. 11.10

From the equality of the arc lengths $\overset{\frown}{AQ}$ and $\overset{\frown}{CQ}$ of the circles, it follows that the angle AOQ equals rt. Let E, F be the orthogonal projections of point P onto the x-axis and y-axis, respectively. Then, for the coordinates x, y of point C we have

$$x = OB - OE = OP \cdot \cos\angle POE + PC \cdot \sin\angle CPE =$$

$$= (1+r)\cos(rt) - r \cdot \cos(t+rt),$$

$$y = OF - OD = OP \cdot \sin\angle POE - PC \cdot \cos\angle CPE =$$

$$= (1+r)\sin(rt) - r \cdot \sin(t+rt).$$

Consequently, we have the parametric equations of the epicycloid

$$\begin{cases} x = (1+r)\cos(rt) - r \cdot \cos(t+rt), \\ y = (1+r)\sin(rt) - r \cdot \sin(t+rt). \end{cases}$$

A special case of the epicycloid is the ***cardioid*** – a curve described by a point fixed on a circle rolling along another circle of the same radius (Fig. 11.11).

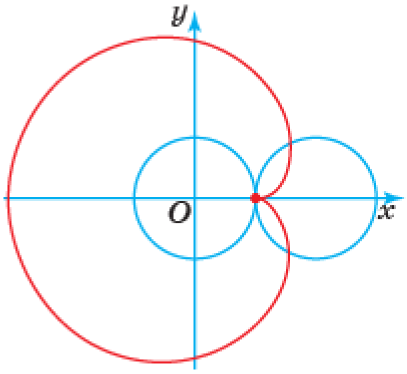


Fig. 11.11

The parametric equations of the cardioid are

$$\begin{cases} x = 2\cos\, t - \cos\, 2t, \\ y = 2\sin\, t - \sin\, 2t. \end{cases}$$

8. Hypocycloid – the trajectory of a point fixed on a circle rolling internally along another circle (Fig. 11.12).

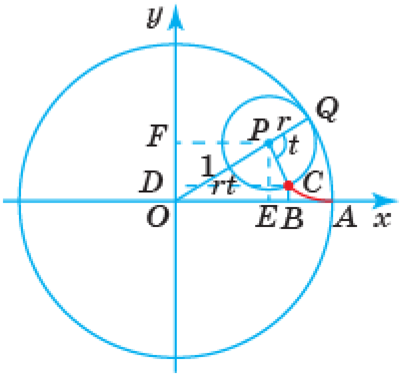


Fig. 11.12

Similarly to the epicycloid, it can be shown that the equations of the hypocycloid are

$$\begin{cases} x = (1-r)\cos(rt) + r\cdot\cos(t - rt), \\ y = (1-r)\sin(rt) - r\cdot\sin(t - rt). \end{cases}$$

A special case of the hypocycloid is the ***astroid*** – the trajectory of a point fixed on a circle rolling internally along another circle whose radius is four times larger (Fig. 11.13).

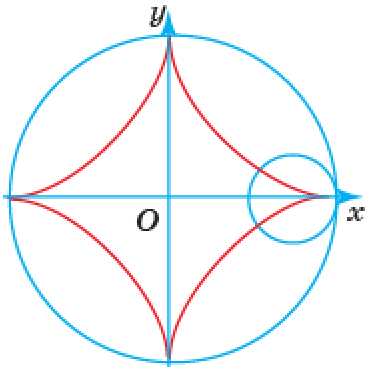


Fig. 11.13

The parametric equations of the astroid are

$$\begin{cases} x = \frac{3}{4}\cos\frac{t}{4} + \frac{1}{4}\cos\frac{3t}{4}, \\ y = \frac{3}{4}\sin\frac{t}{4} - \frac{1}{4}\sin\frac{3t}{4}. \end{cases}$$

These equations can be reduced to the form

$$\begin{cases} x = \cos^3\frac{t}{4}, \\ y = \sin^3\frac{t}{4}. \end{cases}$$

Exercises

1. Write the parametric equations of a circle with center at point $P(x_0, y_0)$ and radius R (Fig. 11.14).

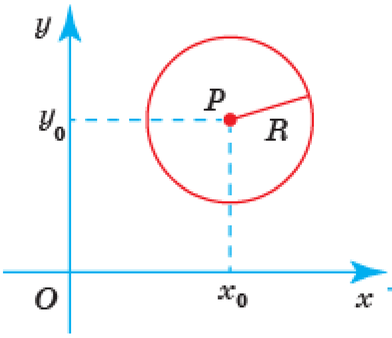


Fig. 11.14

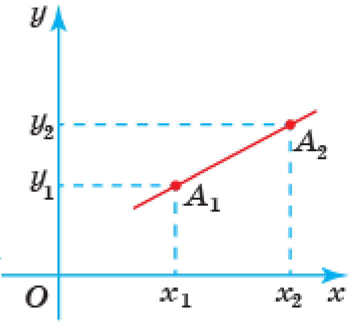


Fig. 11.15

2. Write the parametric equations of a line passing through points $A_1(x_1, y_1)$, $A_2(x_2, y_2)$ (Fig. 11.15).

3. Write the parametric equations of motion along a line forming an angle of 60° with the x-axis, where at the initial moment the point is at the origin, and the speed of the point is 10.

4. Write the parametric equations of motion of a point along a line passing through points $A_1(x_1, y_1)$, $A_2(x_2, y_2)$, where at time t_1 the point is at position A_1, and at time t_2 it is at position A_2.

5. In the GeoGebra computer program, obtain an ellipse whose semi-axes are a and b (Fig. 11.16) by defining it with parametric equations.

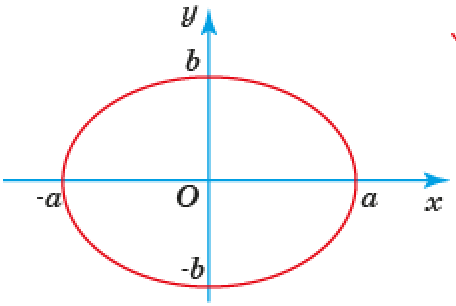


Fig. 11.16

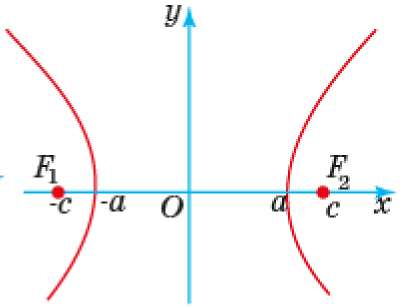


Fig. 11.17

6. In the GeoGebra computer program, obtain a hyperbola (Fig. 11.17) defined by parametric equations.

7. In the GeoGebra computer program, obtain the folium of Descartes (Fig. 11.18) by defining it with the parametric equations $\begin{cases} x = \frac{3at}{1+t^3}, \\ y = \frac{3at^2}{1+t^3}. \end{cases}$

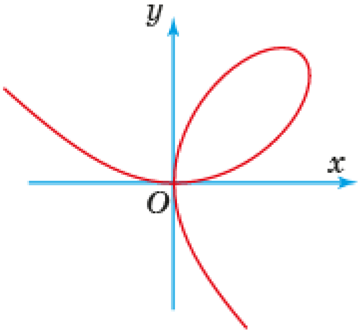


Fig. 11.18

8. Write the parametric equations of a prolate cycloid (Fig. 11.19). Obtain the prolate cycloid in the computer program.

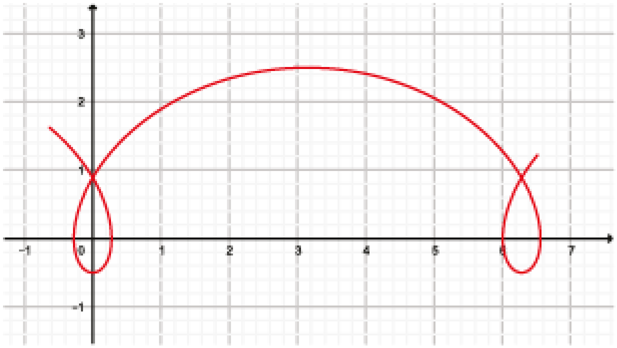


Fig. 11.19

9. In the GeoGebra computer program, obtain the motion of a point along a prolate cycloid.

10. Write the parametric equations of a curtate cycloid (Fig. 11.20). Obtain the curtate cycloid in the GeoGebra computer program.

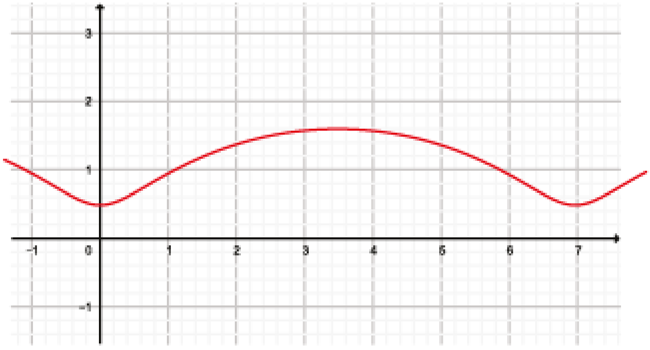


Fig. 11.20

11. In the GeoGebra computer program, obtain the motion of a point along a curtate cycloid.

12. In the GeoGebra computer program, obtain a cardioid (Fig. 11.11).

13. In the GeoGebra computer program, obtain the motion of a point along a cardioid.

14. In the GeoGebra computer program, obtain an epicycloid for which the radius r of the rolling circle equals $\frac{1}{3}$.

15. In the GeoGebra computer program, obtain an epicycloid for which the radius r of the rolling circle equals $\frac{2}{5}$.

16. In the GeoGebra computer program, obtain an epicycloid for which the radius r of the rolling circle equals 2.

17. In the GeoGebra computer program, obtain a hypocycloid for which the radius r of the rolling circle equals $\frac{1}{3}$.

18. In the GeoGebra computer program, obtain a hypocycloid for which the radius r of the rolling circle equals $\frac{1}{4}$.

19. In the GeoGebra computer program, obtain a hypocycloid for which the radius r of the rolling circle equals $\frac{2}{5}$.

20. Using the parametric equations of the astroid (Fig. 11.13), write the equation of the astroid in Cartesian coordinates.

21. Write the parametric equations of a prolate cardioid (Fig. 11.21). Obtain this curve in the computer program.

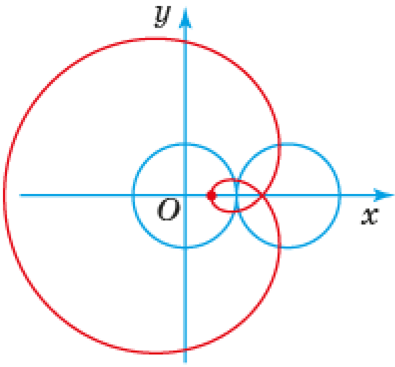


Fig. 11.21

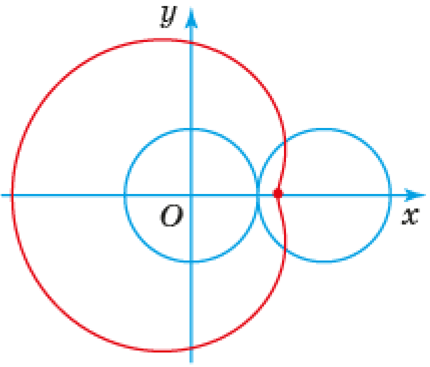


Fig. 11.22

22. Write the parametric equations of a curtate cardioid (Fig. 11.22). Obtain this curve in the GeoGebra computer program.

23. Find the parametric equations of the conchoid of Nicomedes (Fig. 11.23).

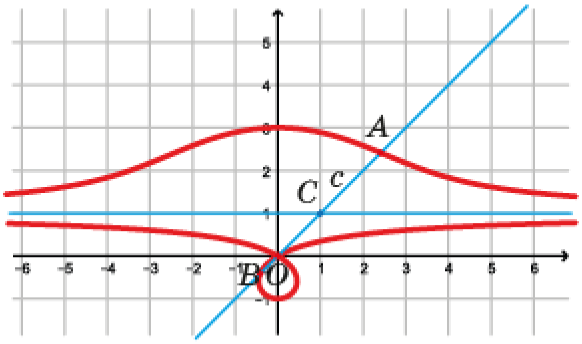


Fig. 11.23

24. Find the parametric equations of the Pascal’s snail (Fig. 11.24).

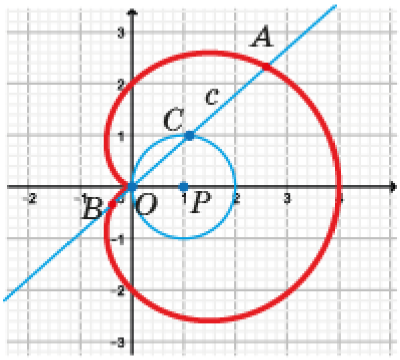


Fig. 11.24

25. Find the parametric equations of the Strophoida (Fig. 11.25).

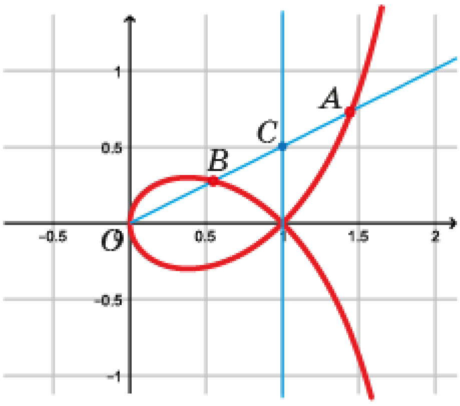


Fig. 11.25

12. Curves defined by equations in polar coordinates

Alongside Cartesian coordinates on a plane, so-called polar coordinates are more convenient in many cases.

Let a coordinate line with a distinguished point O and a unit segment OE be given on a plane. This line is called the polar axis, and the point O is called the pole (Fig. 12.1).

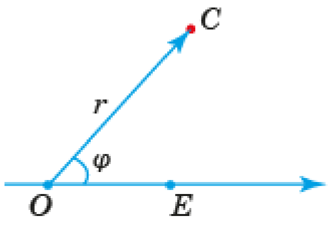


Fig.12.1

The polar coordinates of a point C on a plane with a given polar axis are the pair (r; φ), where r is the distance from point C to point O, and φ is the angle between the polar axis and the vector $\overrightarrow{OC}$, measured counterclockwise if $\varphi > 0$, and clockwise if $\varphi < 0$.

Here, the first coordinate r is called the polar radius, and the second φ is called the polar angle. The polar angle φ can be specified in degrees or radians. To distinguish the notation of polar and Cartesian coordinates, we will denote polar coordinates with a semicolon. This is how polar coordinates are denoted in the GeoGebra computer program, which, along with the Cartesian coordinate system, has a polar coordinate system (Fig. 12.2).

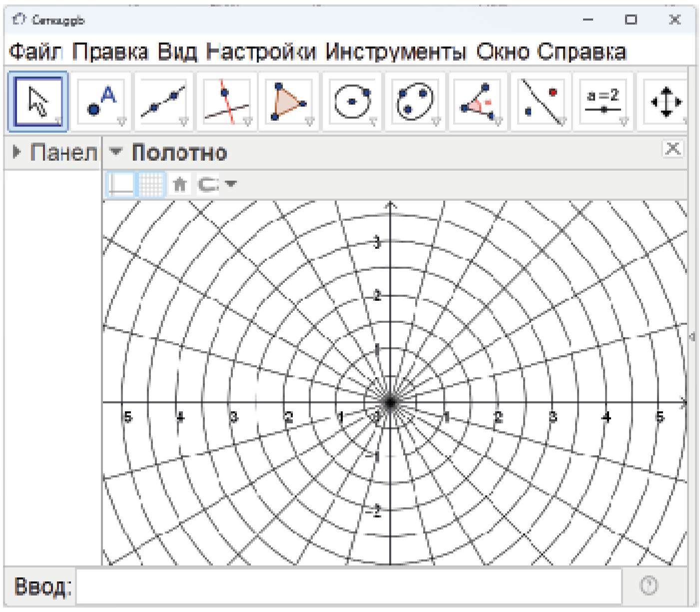


Fig. 12.2

If a Cartesian coordinate system is given on a plane, then the origin is usually taken as the pole, and the Ox axis is taken as the polar axis. In this case, each point on the plane with Cartesian coordinates (x, y) can be assigned polar coordinates (r; φ) (Fig. 12.3).

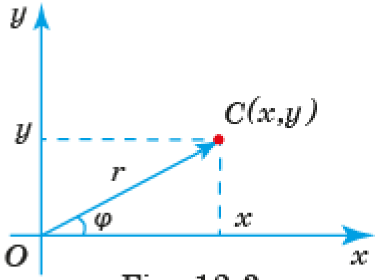


Fig. 12.3

Here, Cartesian coordinates are expressed in terms of polar coordinates by the formulas:

$$\begin{cases} x = r\cos\varphi\,, \\ y = r\sin\varphi\,. \end{cases}$$

Conversely, polar coordinates are expressed in terms of Cartesian coordinates by the formulas:

$$r = \sqrt{x^2 + y^2},\ \cos\varphi = \frac{x}{\sqrt{x^2+y^2}},\ \sin\varphi = \frac{y}{\sqrt{x^2+y^2}}.$$

Polar coordinates turn out to be convenient for defining curves on a plane, especially for defining various spirals (Fig. 12.4). Equations of such curves have the form $r = r(\varphi)$.

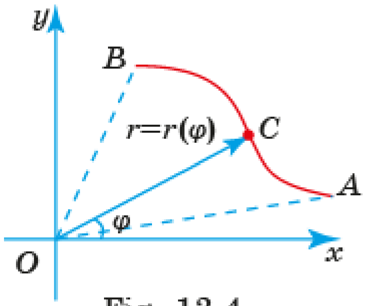


Fig. 12.4

To obtain a curve defined by an equation in polar coordinates in the GeoGebra computer program, you need to type in the "Input" bar: Curve((r(t); t), t, a, b) and press "Enter".

As a result, the desired curve will appear on the screen. Using the formulas connecting polar and Cartesian coordinates, one can transition from the equation of a curve in polar coordinates $r = r(\varphi)$ to the parametric equations of this curve

$$\begin{cases} x = r(t)\cos t\,, \\ y = r(t)\sin t\,. \end{cases}$$

Let us consider examples of curves defined by an equation in polar coordinates.

1. Circle of radius R with center at point O is defined by the equation $r = R$ (Fig. 12.5).

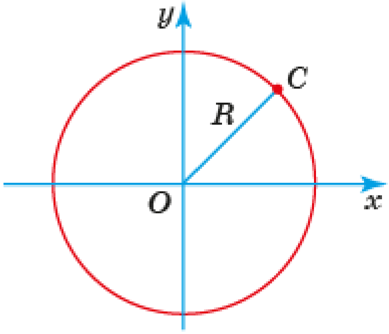


Fig. 12.5

Indeed, a circle is the locus of points at a distance R from point O. All such points satisfy the equality $r = R$. Here, if the angle φ increases, the corresponding point on the circle moves counterclockwise, tracing circles. If the angle φ decreases, the corresponding point traces circles clockwise.

2. Archimedean spiral – a curve defined by the equation $r = a\varphi$, where a is some fixed number, and the angle φ is given in radians. Suppose $a > 0$ and construct the graph of this curve. If $\varphi = 0$, then $r = 0$. This means the curve passes through the origin. Let us see how the radius changes as the angle φ increases. In this case, the radius r will also increase. For example, at $\varphi = \frac{\pi}{2}$ we have $r = \frac{a\pi}{2}$; at $\varphi = \pi$ we get $r = a\pi$, i.e., twice as large. At $\varphi = \frac{3\pi}{2}$, the value of the radius r will be three times larger, and so on. Connecting the obtained points with a smooth curve, we draw the curve called the Archimedean spiral in honor of the scientist who discovered it and studied its properties (Fig. 12.6).

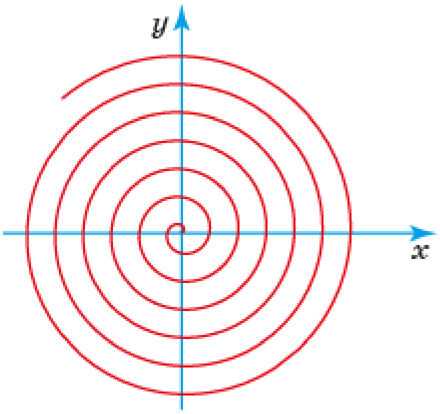


Fig. 12.6

A geometric property characterizing the Archimedean spiral is the constancy of the distances between adjacent turns, each equal to $2\pi a$. Indeed, if the angle φ increases by 2π, i.e., the point makes one revolution counterclockwise, the radius increases by $2\pi a$, which constitutes the distance between adjacent turns. The sound groove on a vinyl record follows an Archimedean spiral. A tightly rolled roll of paper in profile is also an Archimedean spiral. A metal plate with a profile in the shape of half a turn of an Archimedean spiral is often used in variable capacitors. One part of a sewing machine – the mechanism for evenly winding thread onto a bobbin – has the shape of an Archimedean spiral.

3. Golden spiral is defined by the equation in polar coordinates $r = a^{\varphi}$, where a is some fixed positive number other than one, and φ is the angle measured in radians (Fig. 12.7).

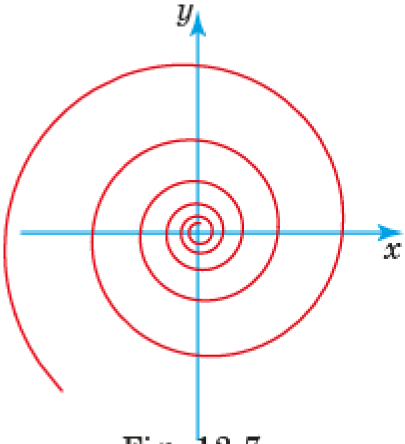


Fig. 12.7

Unlike the Archimedean spiral, the golden spiral is infinite in both directions, since the angle φ can vary from -∞ to +∞. Here, if $a > 1$, as the angle increases, the radius increases; if $0 < a < 1$, as the angle increases, the radius decreases.

A geometric property of this spiral is that each successive turn is similar to the previous one. Indeed, if the angle increases by 2π, i.e., the point makes one revolution counterclockwise, the radius increases by a factor of $a^{2\pi}$. This means that the next turn is similar to the previous one, and the similarity factor is $a^{2\pi}$.

Using this property, after constructing one turn of the golden spiral, all remaining turns can be obtained by similarity.

This property of the golden spiral is used in various technical devices. For example, in the manufacture of rotating knives, which allows maintaining a constant cutting angle during rotation. In hydraulic engineering, pipes that direct water flow to turbine blades are bent along a golden spiral, due to which the water pressure is used with maximum efficiency.

Moths, orienting themselves by parallel moonbeams, instinctively maintain a constant angle between their flight direction and the light ray. However, if they orient themselves to a nearby light source instead of the moon, such as a candle flame, their instinct fails them. Maintaining a constant angle between their flight direction and the light source, they move along a coiling golden spiral and fall into the candle flame.

The shells of many mollusks, snails (Fig. 12.8).

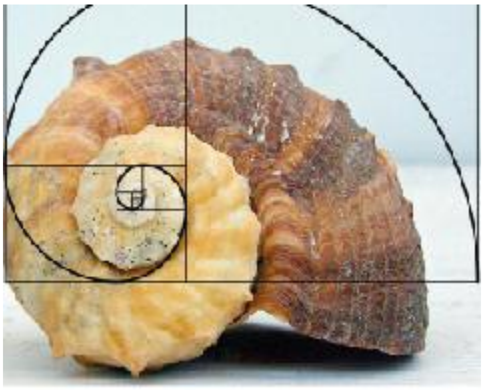


Fig. 12.8

The horns of argali (mountain goats) are twisted in a golden spiral (Fig. 12.9).

Fig. 12.9

The leaves of some plants are also arranged along a golden spiral (Fig. 12.10).

Fig. 12.10

Many galaxies are also twisted along this spiral, in particular the Galaxy of our Solar System (Fig. 12.11).

Fig. 12.11

4. Three-leaf rose – a curve defined by the equation $r = \sin 3\varphi$. To construct this curve, first note that since the radius is non-negative, the inequality $\sin 3\varphi \geq 0$ must hold; solving it we find the range of permissible angles φ: $0° \leq \varphi \leq 60°$; $120° \leq \varphi \leq 180°$; $240° \leq \varphi \leq 300°$.

So, let $0° \leq \varphi \leq 60°$. If the angle φ varies from 0° to 30°, $\sin 3\varphi$ varies from zero to one, therefore the radius r varies from zero to one. If the angle varies from 30° to 60°, the radius varies from one to zero. Thus, as the angle φ varies from 0° to 60°, the point on the plane describes a curve resembling the outline of a petal and returns to the origin. The same petals are obtained when the angle varies within the ranges from 120° to 180° and from 240° to 300° (Fig. 12.12).

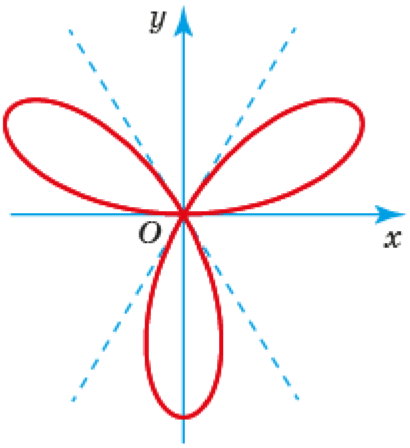


Fig. 12.12

5. Cardioid. Let us prove that the cardioid can be defined by the equation $r = 2(1 - \cos\varphi) = 4\sin^2\frac{\varphi}{2}$.

Consider a unit circle with center at point $Q(-1, 0)$ and a unit circle rolling along it, whose marked point at the initial moment is located at the origin $O(0, 0)$. Suppose the rolling circle rotates around the center of the fixed circle by an angle φ. During this, its center moves to position P, and the marked point moves to position C (Fig. 12.13).

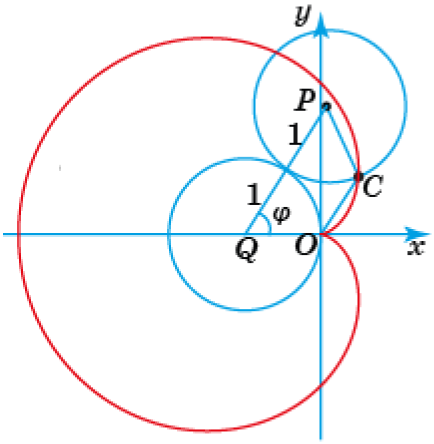


Fig. 12.13

Quadrilateral *OCPQ* is an isosceles trapezoid with base $QP = 2$ and lateral sides $OQ = CP = 1$. Then for the base *OC* we have: $OC = 2 - 2\cos\varphi$. Since $OC = r$, the equation of the cardioid in polar coordinates will be $r = 2(1 - \cos\varphi)$.

6. Conchoid of Nicomedes. Let us recall how it is obtained. Draw a line *c*. Mark a point *O* at a distance d from this line. For an arbitrary point C of line c, draw line *OC*. On it, lay off segments $AC = BC = l$. The locus of points *A* and *B* forms a curve called the conchoid of Nicomedes (Fig. 12.14).

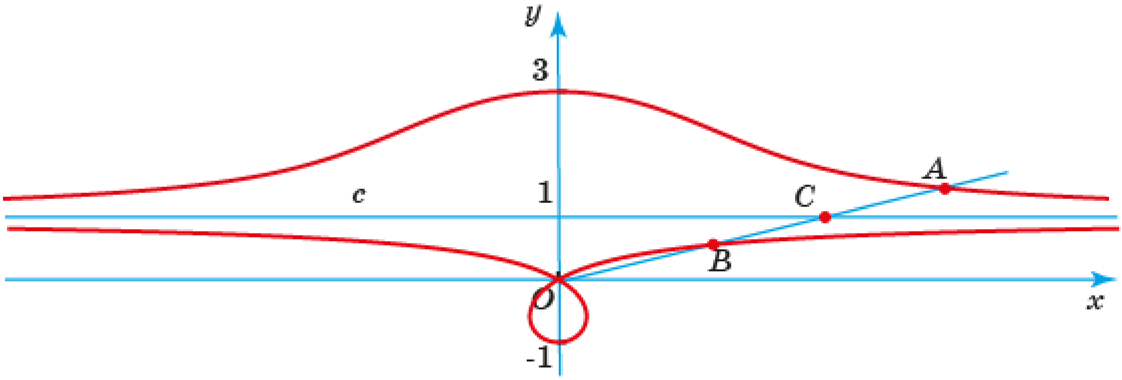


Fig. 12.14

To find the equation of the conchoid in polar coordinates, take point *O* as the origin. Take as line *c* a line parallel to the *x*-axis and at a distance *d* from the origin. Take segments *CA* and *CB* equal to l. Then the desired equations of the conchoid will be

$$r = \frac{d}{\sin\varphi} \pm l.$$

Figure 4.58 shows the conchoid for which $d = 1$, $l = 2$.

7. Limaçon of Pascal. Let us recall how it is obtained. Draw a circle with center *O* and radius *R*. Mark a point *P* on it. For an arbitrary point *C* of this circle, draw line *PC*. On it, lay off segments $AC = BC = l$. The locus of points *A* and *B* forms a curve called the limaçon of Pascal (Fig. 12.15).

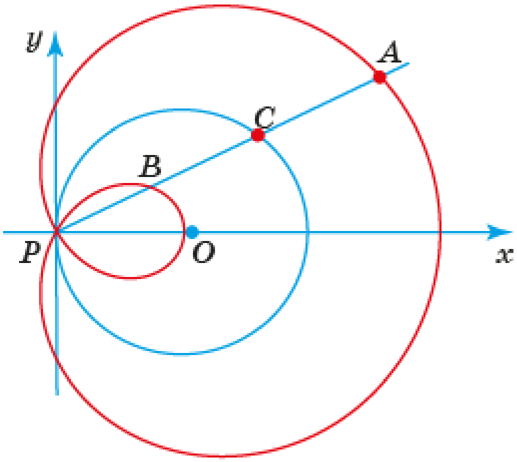


Fig. 12.15

To find the equation of the limaçon of Pascal in polar coordinates, take point P as the origin. Take as the circle a circle of radius R with center at point $O(R, 0)$. Take segments CA and CB equal to l. Then the desired equations of the limaçon will be

$$r = 2R\cos\ \varphi \pm l.$$

Exercises

1. For points with given polar coordinates, find their Cartesian coordinates: a) $(1; \frac{\pi}{3})$; b) $(2; -\frac{\pi}{4})$.

2. For points with given Cartesian coordinates, find their polar coordinates: a) $(\sqrt{2}, \sqrt{2})$; b) $(-10, 0)$; c) $(1, -\sqrt{3})$; d) $(-\sqrt{3}, 1)$.

3. Draw the Archimedean spiral defined by the equation $r = \frac{1}{2\pi}\varphi$.

4. In the GeoGebra computer program, obtain the Archimedean spiral defined by the equation $r = \frac{1}{2\pi}\varphi$.

5. Write the parametric equations of the Archimedean spiral $r = a\varphi$.

6. Draw the golden spiral defined by the equation $r = 1{,}1^{\varphi}$.

7. In the GeoGebra computer program, obtain the golden spiral defined by the equation $r = 1{,}1^{\varphi}$.

8. Write the parametric equations of the logarithmic spiral.

9. In the GeoGebra computer program, obtain the curve defined by the equation $r = \sin 5\varphi$.

10. Draw the curve defined by the equation $r = \cos\ \varphi$. Identify its type. Obtain it in the GeoGebra computer program.

11. Draw the curve defined by the equation $r = \sin\ \varphi$. Identify its type. Obtain it in the GeoGebra computer program.

12. Draw the curve defined by the equation $r = \frac{1}{\cos\varphi}$. Identify its type. Obtain it in the GeoGebra computer program.

13. Draw the curve defined by the equation $r = \frac{1}{\sin\varphi}$. Identify its type. Obtain it in the GeoGebra computer program.

14. Draw the hyperbolic spiral – a curve defined by the equation $r = \frac{1}{\varphi}$.

15. Draw the Galilean spiral, which is defined by the equation $r = a\varphi^2$ ($a > 0$). Obtain it in the GeoGebra computer program.

16. In the GeoGebra computer program, obtain the curve defined by the equation $r = \sin\frac{5\varphi}{3}$.

17. In the GeoGebra computer program, obtain the cochleoid, which is defined by the equation $r = \frac{\sin\varphi}{\varphi}$.

18. In the GeoGebra computer program, obtain the curve defined by the equation $r = 1 + \cos 3\varphi + \sin^2 3\varphi$.

19. In the GeoGebra computer program, obtain the curve defined by the equation $r = 30 + 15\sin(60\varphi)\sin(2{,}5\varphi)$.

20. Find the equation of a parabola in polar coordinates, taking the focus F of the parabola, which is at a distance $2a$ from the directrix d, as the pole, and the axis of the parabola as the polar axis (Fig. 12.16).

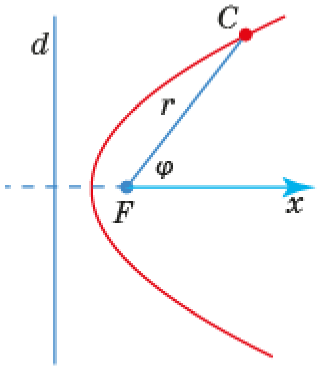


Fig. 12.16

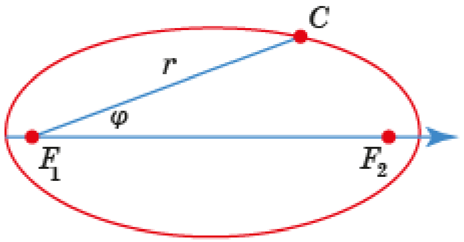


Fig. 12.17

21. Find the equation of an ellipse in polar coordinates, taking the focus F_1 of the ellipse as the pole, and the line containing the major axis of the ellipse as the polar axis (Fig. 12.17). Take the distance between the foci as $2c$, and the length of the major axis as $2a$.

22. Find the equation of a hyperbola in polar coordinates, taking the focus F_2 of the hyperbola as the pole, and the line containing the axis of the hyperbola as the polar axis (Fig. 12.18). Take the distance between the foci as $2c$, and the length of the axis as $2a$.

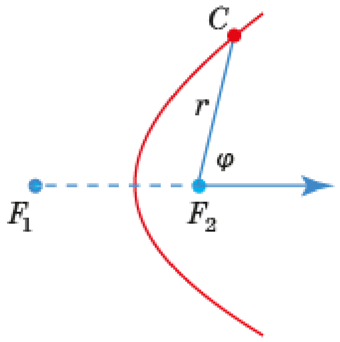


Fig. 12.18

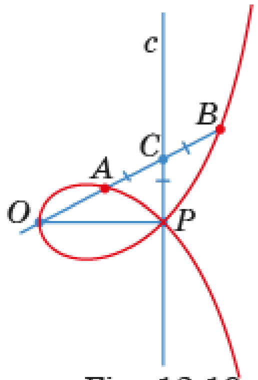


Fig. 12.19

23. Determine which curve is defined by the equation $r = \frac{1}{1-\varepsilon\cdot\cos\varphi}$ if: a) $|\varepsilon| = 1$; b) $0 < |\varepsilon| < 1$; c) $|\varepsilon| > 1$. Obtain it in the GeoGebra computer program.

24. Let us recall how the strophoid is obtained. Points $O(0, 0)$, $P(d, 0)$ and line c defined by the equation $x = d$ are given. Lines are drawn through point O at angles φ to the x-axis, intersecting line c at points C. From points C on these lines, segments $CA = CB = CP$ are laid off (Fig. 12.19). The curve described by points A and B as the angles φ vary is the strophoid. Find its equation in polar coordinates. Obtain it in the GeoGebra computer program.

25. Let us recall how the cissoid of Diocles is obtained. Points $P(0, 0)$, $Q(d, 0)$, a circle with diameter PQ, and line b defined by the equation $x = d$ are given. Rays are drawn through point P at angles φ to the x-axis, intersecting the circle at points A and line b at points B. From point P on these rays, segments $PC = AB$ are

laid off (Fig. 12.20). The curve described by points C is the cissoid of Diocles. Find its equation in polar coordinates. Obtain it in the GeoGebra computer program.

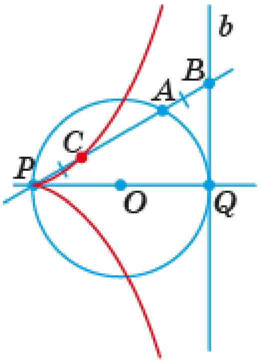


Fig. 12.20

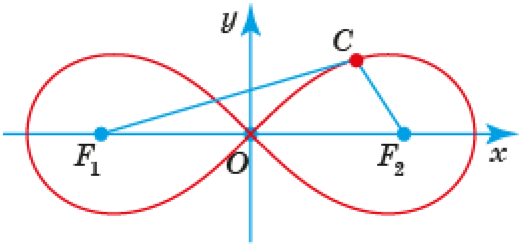


Fig. 12.21

26. Let us recall that the lemniscate of Bernoulli is defined by the equation $(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$ (Fig. 12.21). Find its equation in polar coordinates.

27. Let us recall how the Maclaurin trisectrix is obtained. Two points $A(0, 0)$ and $B(d, 0)$ are given. Line AB rotates around point A by an angle φ counterclockwise and around point B by an angle 3φ counterclockwise. The locus of intersection points of the resulting lines is the Maclaurin trisectrix (Fig. 12.22). Find its equation in polar coordinates. Obtain it in the GeoGebra computer program.

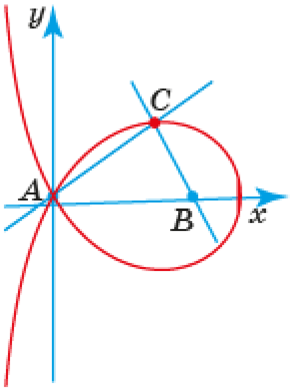


Fig. 12.22

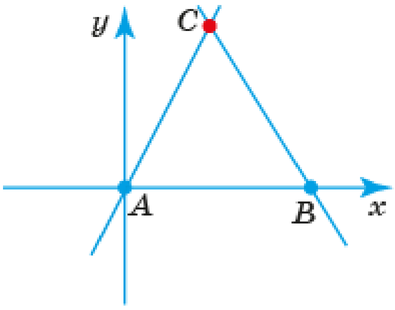


Fig. 12.23

28. Two points $A(0, 0)$ and $B(d, 0)$ are given. Line AB rotates around point A by an angle φ counterclockwise and around point B by an angle 2φ counterclockwise (Fig. 12.23). Find the equation in polar coordinates of the locus of intersection points of the resulting lines. Obtain it in the GeoGebra computer program.

29. Recall that an inversion with center O and radius R is a transformation of the plane such that each point A in the plane, distinct from O, is mapped to a point A' lying on the ray OA and satisfying $OA \cdot OA' = R^2$ (Fig. 12.24). Using the equation of the strophoid in polar coordinates, prove that under inversion with respect to a circle with center O and radius d, the strophoid (Fig. 12.19) maps onto itself.

30. Using polar coordinates determine what curve is the image of the parabola given by the equation $4ax = y^2$ under inversion with center $F(a, 0)$ (Fig. 12.25). Obtain this curve using the computer program GeoGebra.

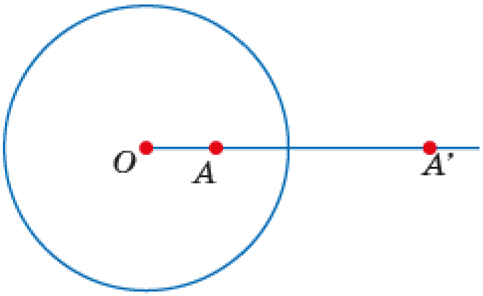


Fig. 12.24

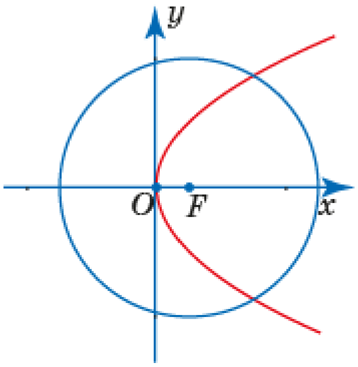


Fig. 12.25

31. Using polar coordinates determine what curve is the image of the parabola given by the equation $4ax = y^2$ under inversion with center $O(0,0)$ (Fig. 12.26). Obtain this curve using the computer program GeoGebra.

32. Using polar coordinates determine what curve is the image of the hyperbola given by the equation $x^2 - y^2 = 1$ under inversion with center $O(0,0)$ and radius 1 (Fig. 12.27). Obtain this curve using the computer program GeoGebra.

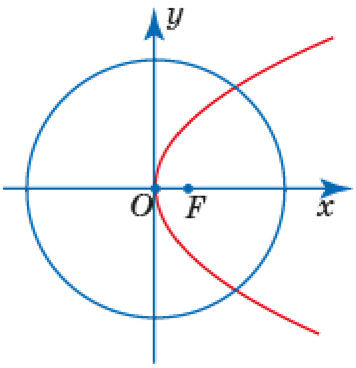


Fig. 12.26

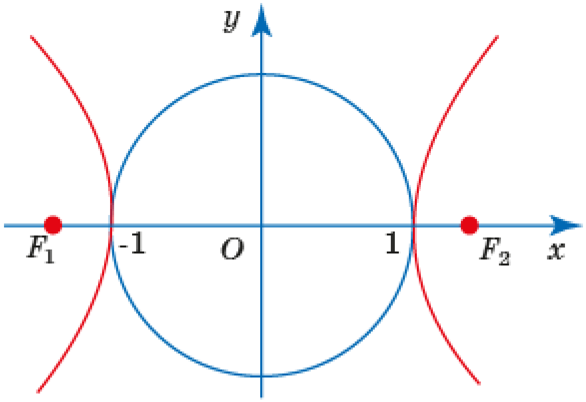


Fig. 12.27

33. Using the computer program GeoGebra determine what curve is the image of the hyperbola given by the equation $x^2 - y^2 = 1$ under inversion with center $P(1,0)$ and radius 2 (Fig. 12.28).

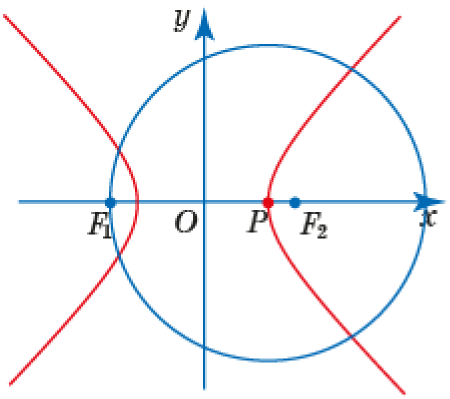


Fig. 12.28

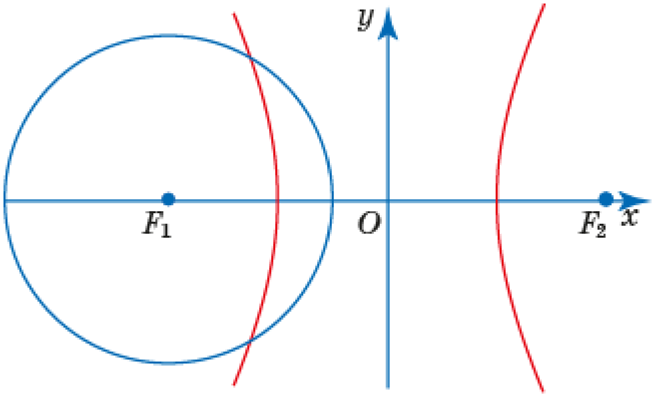


Fig. 12.29

34. Using the computer program GeoGebra determine what curve is the image of the hyperbola with foki F_1, F_2 under inversion with center F_1 and radius R (Fig. 12.29).

IV*. CURVES ON THE SPHERE AND THE POINCARE MODEL OF THE LOBACHEVSKY PLANE

13. Curves on the sphere

Consider a sphere with center O and radius R.

Great circles on the sphere, i.e., circles formed by the intersection of the sphere with planes passing through the center of the sphere, will be called ***spherical lines*** (Fig. 13.1).

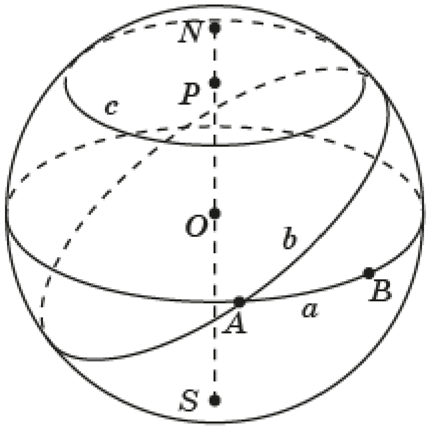


Fig. 13.1

Arcs of great circles will be called ***spherical segments***.

The length of a spherical segment will be called the length of the corresponding arc.

The spherical distance between two points on the sphere will be called the length of the shortest spherical segment connecting these points.

The angle between two spherical lines will be called the angle between the tangents drawn through the point of intersection of these lines.

For a given spherical line, consider a line passing through the center of the sphere and perpendicular to the plane of the corresponding circle. The points of intersection of this line with the sphere will be called ***the spherical poles*** of this spherical line.

Circles on the sphere, formed by the intersection of the sphere with planes not passing through the center of the sphere, will be called ***spherical circles***.

For a given spherical circle, consider a line passing through the center of the sphere and perpendicular to the plane of this circle. The point of intersection of this line with the sphere will be called ***the center*** of the spherical circle. The spherical distance from this center to the points on the spherical circle will be called ***the radius*** of the spherical circle.

Parabola. Consider a spherical line d and a point F not belonging to this line. Let N and S denote the corresponding poles. We will find the spherical analog of a parabola, i.e., the locus of points A on the sphere that are equidistant from point F and the spherical line d (Fig. 13.2).

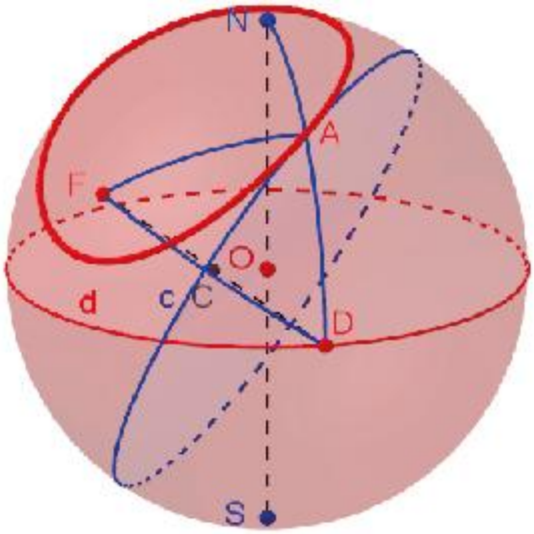


Fig. 13.2

Consider any point D on a spherical line d. Connect it with a spherical segment to point F. Let C be the midpoint of this segment. Through point C, draw a plane perpendicular to the spherical line FD. The spherical line c, which is the intersection of this plane with the sphere, will be the locus of points equidistant from points F and D. Through point D draw a spherical line b, which is perpendicular to the line d. The point A, the intersection of the spherical lines b and c, will be equidistant from point F and the spherical line d. As the position of point D changes on the spherical line d, point A will trace the desired locus of points. We will call this a ***spherical parabola***, and the spherical line d will be the directrix. The spherical line c will be ***the tangent***, and point A will be ***the point of tangency***.

The indicated construction of point A can be performed in the GeoGebra computer program. If the option "Trace On" is selected for the obtained point A, then as point D moves along the spherical line d, point A will trace the desired spherical parabola.

Note that from the equality of arcs $\breve{AF}$ and $\breve{AD}$, it follows that the sum of the lengths of arcs $\breve{AD}$ and $\breve{AN}$ is equal to the length of arc $\breve{DN}$, i.e., equal to $\frac{R\pi}{2}$ and independent of the position of point D on the spherical line d. Consequently, the constructed locus of points on the sphere can also be considered an analog of an ellipse on a plane.

If point F coincides with one of the poles of the spherical line d, the spherical parabola will be a spherical circle centered at that pole and radius $\frac{R\pi}{4}$.

Ellipse. Consider two points F_1, F_2 on a sphere and a positive number c, greater than the distance between these points and less than $R\pi$. We will find the spherical analog of an ellipse with foci F_1, F_2 and constant c, i.e., the locus of points on the sphere such that the sum of their distances to F_1 and F_2 is equal to c.

Consider a spherical circle d with the center F_2 and the radius c. Let D be a point on this circle. Connect it with a spherical segment to point F_1. Let C be the midpoint of this segment. Through point C, draw a plane perpendicular to the spherical line F_1D. The spherical line c, which is the intersection of this plane with the sphere, will be the locus of points equidistant from points F_1 and D. Through point D draw a spherical line b, which is perpendicular to the circle d. The point A, the intersection of spherical lines b and c, will be equidistant from point F_1 and the spherical circle d (Fig. 13.3). The sum of the distances from the point A to F_1 and F_2 will be equal to $\overset{\frown}{F_2D} = c$. As the position of point D changes on the spherical circle d, point A will trace the desired locus of points. We will call this a ***spherical ellipse***, and the points F_1, F_2 will be the foci. The spherical line c will be ***the tangent***, and point A will be ***the point of tangency***.

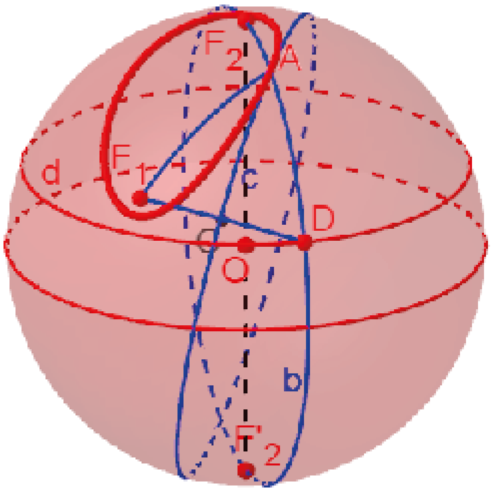


Рис. 13.3

The indicated construction of point A can be performed in the GeoGebra computer program. If the option "Trace On" is selected for the obtained point A, then as point D moves along the spherical circle d, point A will trace the desired spherical ellipse.

Hyperbola. Consider two points F_1, F_2 on a sphere and a positive number c, lesser than the distance between these points. We will find the spherical analog of a hyperbola with foci F_1, F_2 and constant c, i.e., the locus of points on the sphere such that the absolute difference of their distances to F_1 and F_2 is equal to c.

Consider a spherical circle d with the center F_2 and the radius c. Let D be a point on this circle. Connect it with a spherical segment to point F_1. Let C be the midpoint of this segment. Through point C, draw a plane perpendicular to the spherical line F_1D. The spherical line c, which is the intersection of this plane with the sphere, will be the locus of points equidistant from points F_1 and D. Through point D draw a spherical line b, which is perpendicular to the circle d. The point A the intersection of lines b and c, will be equidistant from point F_1 and the spherical circle d (Fig. 13.4). The difference of the distances from the point A to F_2 and F_1 will be equal to $\overset{\frown}{F_2D} = c$. As the position of point D changes on the spherical circle d, point A will trace the arc of the hyperbola. The spherical line c will be ***the tangent***, and point A will be ***the point of tangency***.

Let A' be the point symmetric to A with respect to O. Then $\overset{\frown}{A'F_1} - \overset{\frown}{A'F_2} = \frac{R\pi}{2} - \overset{\frown}{AF_1} - \overset{\frown}{AF_2} = \overset{\frown}{F_2D} = c$. As the position of point D changes on the spherical circle d, point A' will trace the second arc of the hyperbola.

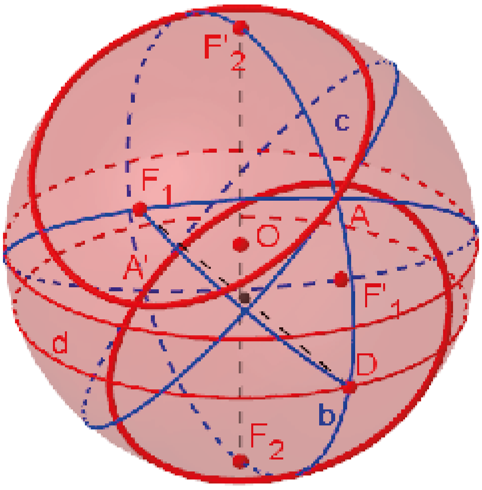


Рис. 13.4

Let F'_1, F'_2 be the points symmetric correspondingly to F_1, F_2 with respect to O. Then the first arc of the hyperbola is the ellipse with foci F_1, F'_2. The second arc of the hyperbola is the ellipse with foci F_2, F'_1.

The indicated construction of points A, A' can be performed in the GeoGebra computer program. If the option "Trace On" is selected for the obtained points A, A', then as point D moves along the spherical circle d, points A, A' will trace the desired spherical hyperbola.

Exercises

1. Construct a spherical parabola using the GeoGebra computer program.

2. For a given focus F and directrix d of a spherical parabola construct the tangent passing through a given point A (Fig. 13.5).

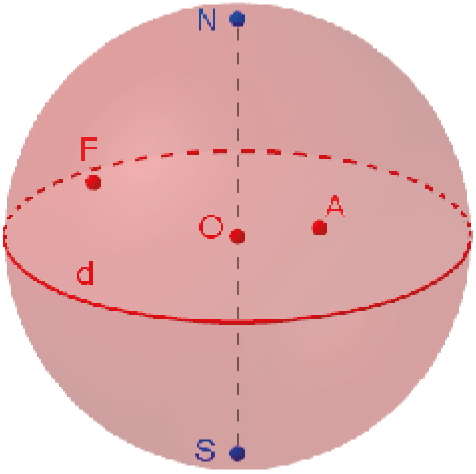


Рис. 13.5

3. Prove that the tangents to a spherical parabola, passing through a point A on the directrix d, are perpendicular (Fig. 13.6).

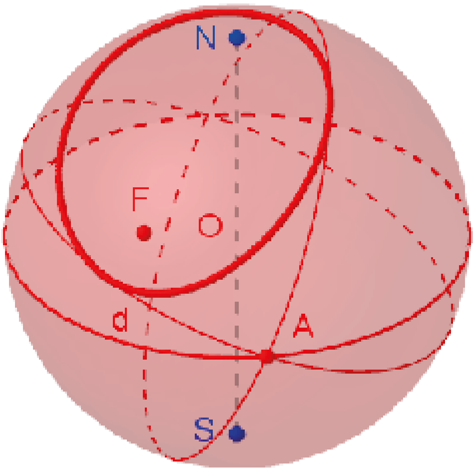


Рис. 13.6

4. Construct a spherical ellipse using the GeoGebra computer program.

5. For a given foci F_1, F_2 and the constant c of a spherical ellipse construct the tangent passing through a given point A (Fig. 13.7).

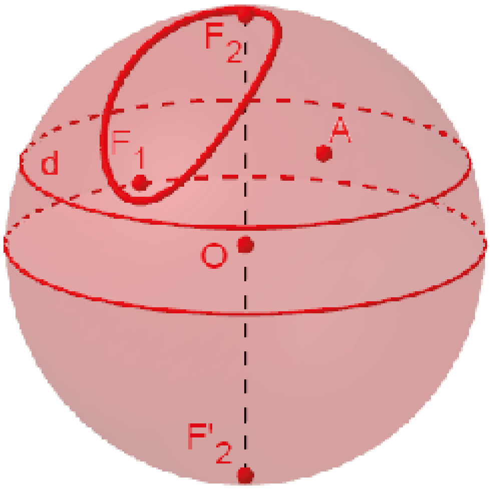


Рис. 13.7

6. Construct a spherical hyperbola using the GeoGebra computer program.

7. For a given foci F_1, F_2 and the constant c of a spherical hyperbola construct the tangent passing through a given point A (Fig. 13.8).

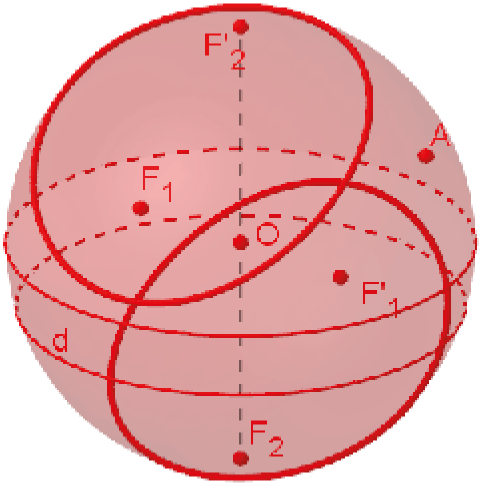


Рис. 13.8

8. Using the computer program GeoGebra, construct a spherical analogue of the conchoid of Nicomedes.

9. Using the computer program GeoGebra, construct a spherical analogue of the Pascal’s snail.

14. Curves on the Poincare model of the Lobachevsky plane

One of the models of Lobachevsky geometry was proposed by the French mathematician H. Poincaré (1854-1912).

In this model, the Lobachevsky plane is the interior of a disk. We will simply call it the ***Lobachevsky plane***. The circle itself and the circumference bounding it will be called the Poincaré disk and the Poincaré circle, respectively. Points in the Lobachevsky plane are points located inside the Poincaré disk. Lines are diameters of the Poincaré disk excluding their endpoints, as well as arcs of circles located inside the Poincaré disk and perpendicular to its circumference (Fig. 14.1). We will call these Lobachevsky lines.

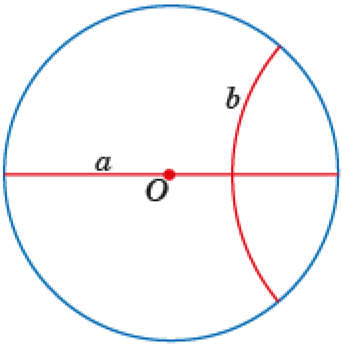


Fig. 14.1

To construct a Lobachevsky line, we will use the computer program GeoGebra. Mark any point P outside the Poincaré disk. Using the "Tangent" tool, draw tangents through point P. Using the "Intersection" tool, find their points of tangency, A and B. With the "Arc through center and two points" tool, construct an arc of a circle with center P and endpoints A and B. This will be the desired hyperbolic line AB (Fig. 14.2).

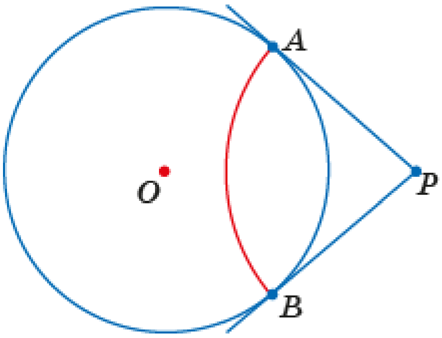


Fig. 14.2

For other constructions, we will need the inversion transformation and some of its properties.

Recall that inversion with respect to a circle with center O and radius R maps each point A in the plane, other than O, to a point A' lying on the ray OA such that $OA \cdot OA' = R^2$.

Under inversion, points A located inside the circle are mapped to points A' located outside the circle, and vice versa. Points on the circle remain in place.

Let's construct the inversion of point A, located outside a circle with center O and radius R. Through point A, we draw a tangent to the circle bounding this disk. Let B be the point of tangency. From this point, we drop a perpendicular BA' to the line OA (Fig. 14.3).

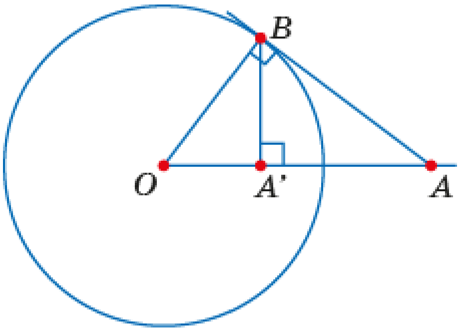


Fig. 14.3

Triangles $OA'B$ and OBA are similar. Consequently, the equality $\frac{OA'}{OB} = \frac{OB}{OA}, =$ holds, from which we obtain the equality $OA' \cdot OA = R^2$. Therefore, point A' is the inversion of point A.

Inversion can be obtained in the computer program GeoGebra, which has the "Reflect Object in Circle" tool for this purpose.

We will use the following properties of inversion, the proofs of which can be found in the book by Bakelman, I. Ya. *Inversion*. Moscow: Nauka, 1966.

Property 1. Inversion with respect to a given circle transforms a line not passing through its center into a circle passing through the center, excluding the center itself, and vice versa.

Property 2. Inversion with respect to a given circle transforms a circle not passing through its center into a circle not passing through the center.

The validity of these properties can be verified in the computer program GeoGebra.

Let's consider some additional properties of inversion.

Property 3. A circle perpendicular to a given circle is mapped onto itself under inversion.

Proof. Consider a circle with center P that is perpendicular to a circle with center O (Fig. 14.4).

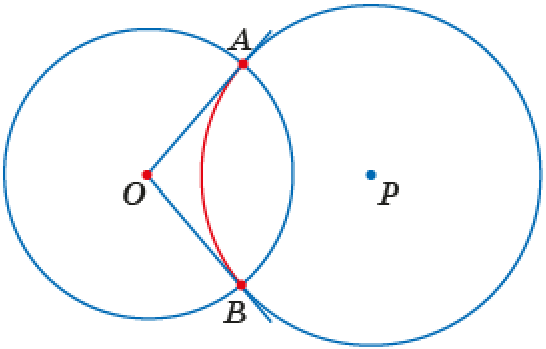


Fig. 14.4

Let A and B be the points of intersection of these circles. Then the lines OA and OB are tangent to the circle with center P. Since this circle is located between these tangents, the circle obtained by its inversion will also be located between these tangents. As a circle is uniquely determined by two of its points and the tangents drawn through these points, the circle with center P will be mapped onto itself under inversion.

Property 4. A circle passing through two points that are inverses of each other with respect to a given circle is perpendicular to that circle.

Proof. Let point A' be the inverse of point A with respect to circle a with center O, and let circle b with center P pass through these points. Let B be one of the intersection points of circles a and b (Fig. 14.5).

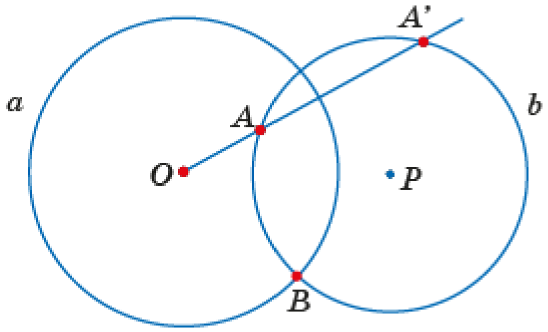


Рис. 14.5

Since a unique circle passes through three points not lying on the same straight line, circle b, under inversion with respect to circle a, maps onto itself. Consequently, it is perpendicular to circle a.

Using these properties, we will construct the Lobachevsky line passing through points A and B in the Lobachevsky plane.

If points A and B lie on the same line with center O, then the desired line is the diameter passing through these points. If points A and B do not lie on a diameter (Fig. 14.6), then we construct the inversion A' of point A. We draw a circle through points A, B, and A'. This circle will be perpendicular to the Poincaré circle, and the arc CD of the constructed circle, located inside the Poincaré disk, will be the desired Lobachevsky line passing through points A and B.

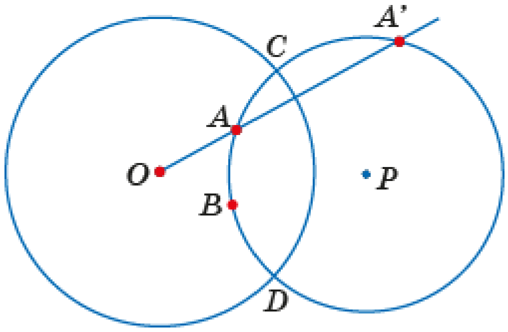


Fig. 14.6

Let's define the concept of congruence of figures in the Lobachevsky plane. To do this, consider the following transformations of the Poincaré disk with center O.

1. Rotation around point O.

2. Axial symmetry with respect to a line c passing through point O.

3. Inversion with respect to a circle c perpendicular to the Poincaré circle.

We will also call this axial symmetry with respect to a Lobachevsky line.

Note that from the condition of perpendicularity of circle c and the Poincaré circle, it follows that under inversion with respect to circle c, points A located on the Poincaré circle are mapped to points A' located on the Poincaré circle, and points A located inside the Poincaré disk are mapped to points A' located inside this disk (Fig. 14.7).

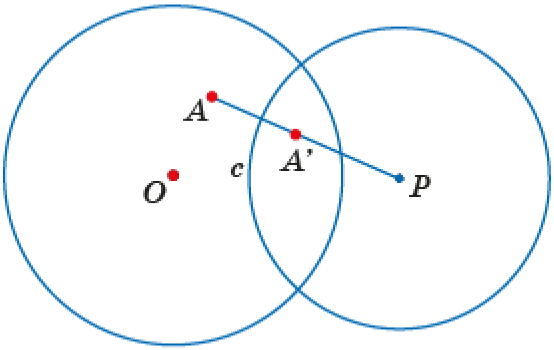


Fig. 14.7

Thus, axial symmetry with respect to a Lobachevsky line is a transformation of the Lobachevsky plane onto itself.

Figures F and F', located on the Lobachevsky plane, will be called equal if they are obtained from each other by means of the transformations indicated above or their compositions.

Property 5. Circles on the Lobachevsky plane are ordinary circles.

Proof. It is clear that a circle on the Lobachevsky plane with center at point O is an ordinary circle (Fig. 14.8).

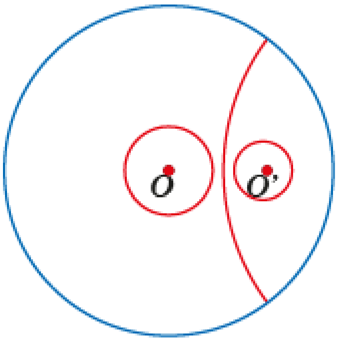


Fig. 14.8

Any other circle is obtained from such a circle by axial symmetry. Considering that inversion transforms ordinary circles into ordinary circles, we get that circles on the Lobachevsky plane are ordinary circles.

Note that the center O' of a circle on the Lobachevsky plane differs from the ordinary center of this circle.

Let's present a few problems on constructing figures in the Lobachevsky plane.

Problem 1. In the Lobachevsky plane, construct a Lobachevsky line perpendicular to a given Lobachevsky line *AB*, passing through a given point *C* not belonging to the given line (Fig. 14.9).

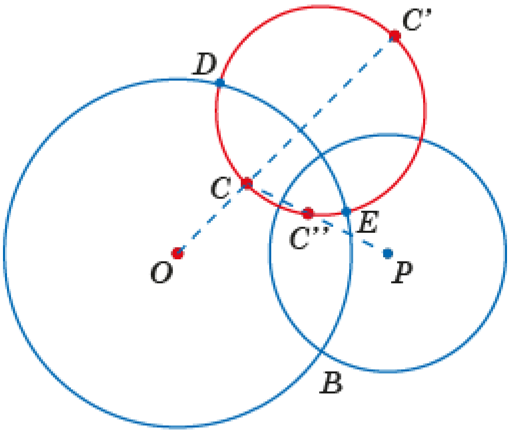


Fig. 14.9

Solution. Construct an inversion *C′* of point *C* with respect to a circle centered at *O*, and an inversion *C″* of point *C* with respect to a circle centered at *P*. Draw a circle through points *C*, *C′*, and *C″*. Its arc *DE* will be the desired Lobachevsky line.

Problem 2. In the Lobachevsky plane, construct the perpendicular bisector to the segment *CD* of the Lobachevsky line *AB* (Fig. 14.10).

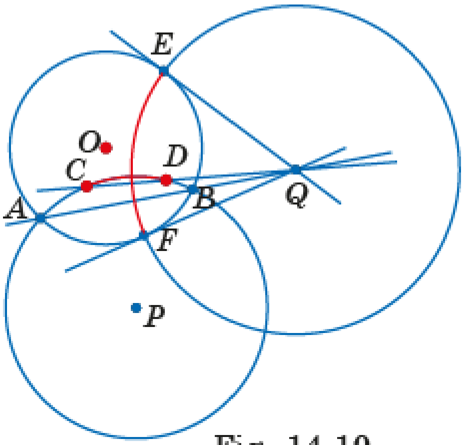


Fig. 14.10

Solution. Since the arc of a circle corresponding to the perpendicular bisector must be perpendicular to the Lobachevsky line *AB*, its center *Q* must lie on the line *AB*. Since, under inversion with respect to this circle, point *C* must map to point *D*, the center *Q* must lie on the line *CD*.

Draw lines *AB* and *CD*. Let *Q* be their intersection point. Through point *Q*, draw tangents to the Poincaré circle. Let *E* and *F* be the points of tangency. With center at point *Q* and radius *QE*, draw a circle. Its arc *EF* will be the desired perpendicular bisector.

Problem 3. On the Lobachevsky plane, construct the bisector of the angle formed by two Lobachevsky lines A_1B_1 and A_2B_2 intersecting at point C (Fig. 14.11).

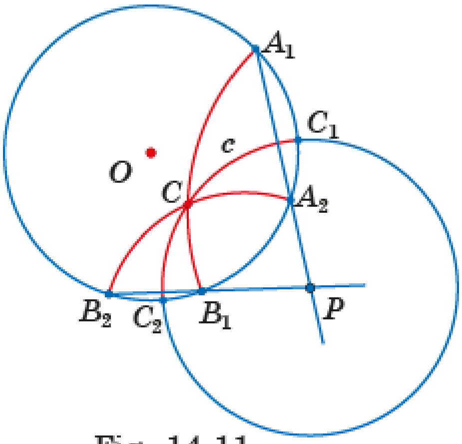


Fig. 14.11

Solution. Draw the line A_1A_2. Since under inversion with respect to the circle $c$corresponding to the desired bisector, point A_1 maps to point A_2, the center P of this circle must lie on the line A_1A_2. Draw the line B_1B_2. The center P of the circle on which the desired bisector $c$lies must be located on this line. Thus, point P is the intersection point of the lines $A_1A_2$and B_1B_2. Draw a circle with center P and radius PC. The desired bisector C_1C_2will lie on this circle.

Parabola. In the GeoGebra computer program, we construct an analogue of a parabola on the Lobachevsky plane. Consider a Lobachevsky line AB (Fig. 14.12).

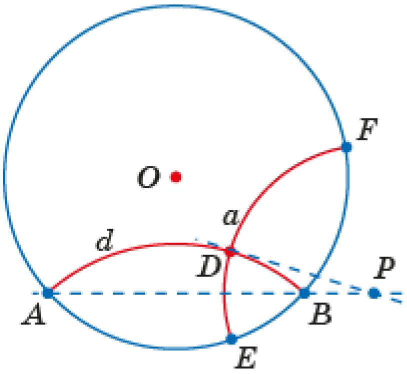


Fig. 14.12

Mark point D on it. Through point D, draw a Lobachevsky line a perpendicular to the Lobachevsky line AB. To do this, draw line AB and the tangent to the arc $\breve{AB}$ passing through point D. Let P denote their intersection point. With center at point P, draw an arc $\breve{EF}$ of a circle passing through point D. This arc will be perpendicular to arc $\breve{AB}$. Consequently, it gives the desired Lobachevsky line a.

Construct the perpendicular bisector b of segment OD. To do this, draw line OD. Through point D, draw a Lobachevsky line perpendicular to line OD. Denote its intersection points with the Poincaré circle as G and H. Through point

G, draw a tangent to the Poincaré circle. Denote its intersection point with line *OD* as *Q*. With center at point *Q*, draw an arc $\overset{\frown}{GH}$ of a circle passing through point *G*. The corresponding Lobachevsky line *b* will be the perpendicular bisector of segment *OD* (Fig. 14.13).

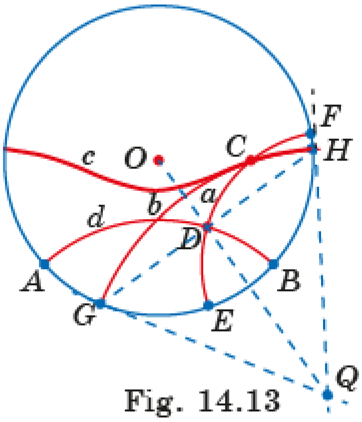


Fig. 14.13

Let C denote the intersection point of the Lobachevsky lines *a* and *b*. It will be equidistant from point *O* and the Lobachevsky line *d*. In the properties of this point, select the "Trace On" option. As point *D* moves along the Lobachevsky line *d*, point *C* will move, leaving a trace in the form of a parabola *c*.

Ellipse. In the GeoGebra computer program, we construct an analogue of an ellipse on the Lobachevsky plane. Consider a Lobachevsky circle *d* with its center at the center *O* of the Poincaré circle and radius *r*. Note that it is an ordinary circle. Mark a point *P* inside it, distinct from *O* (Fig. 14.14).

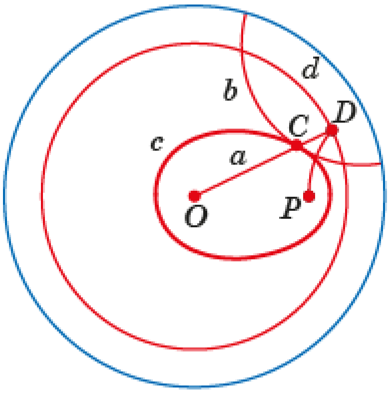


Fig. 14.14

Draw line a through points *O* and *D*. Construct the perpendicular bisector *b* of the Lobachevsky segment *PD*. Denote by *C* the intersection point of lines *a* and *b*. The sum of the distances from *C* to points *O* and *P* will equal *r*. In the properties of point *C*, select the "Trace On" option. As point *D* moves along circle *d*, point *C* will move, leaving a trace in the form of an ellipse.

Гипербола. In the GeoGebra computer program, we construct an analogue of a hyperbola on the Lobachevsky plane. Consider a Lobachevsky circle *d* with its center at the center *O* of the Poincaré circle. Mark a point *P* inside it, distinct from *O* (Fig. 14.15).

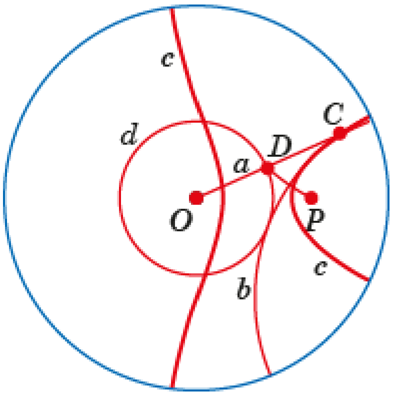


Fig. 14.15

Draw line a through points O and D. Construct the perpendicular bisector b of the Lobachevsky segment PD. Denote by C the intersection point of lines a and b. The difference of the distances from C to points O and P will equal r. In the properties of point C, select the "Trace On" option. As point D moves along circle d, point C will move, leaving a trace in the form of a branch of a hyperbola.

The second branch of the hyperbola is obtained by reflecting this branch across the perpendicular bisector of segment OP.

Regular polygon. On the Lobachevsky plane, we construct a regular pentagon with center at point O, whose angles are equal to 90°.

Construct a Lobachevsky line corresponding to a circle with center P. Rotate this circle around point O by an angle of 72° counterclockwise. Denote by A the intersection point of the original circle and the rotated one. Measure the angle formed by these circles. By moving point P, find its position such that this angle equals 90°. Continuing to rotate the constructed circle around point O by angles of 72°, we obtain circles that bound the regular pentagon $ABCDE$, whose angles are equal to 90° (Fig. 14.16).

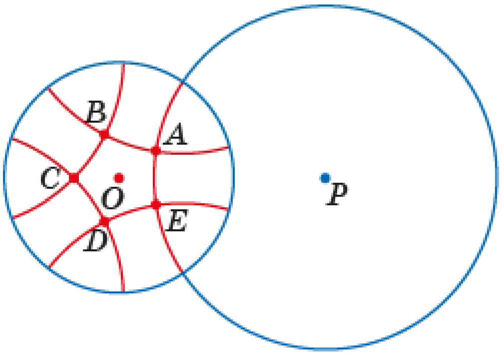


Fig. 14.16

The position of point P can be found exactly. To do this, let us establish the relationship between the angle α through which arc a rotates, the angle φ between arc a and arc b (obtained by rotating arc a by angle α), and the distance d between points O and P (Fig. 14.17).

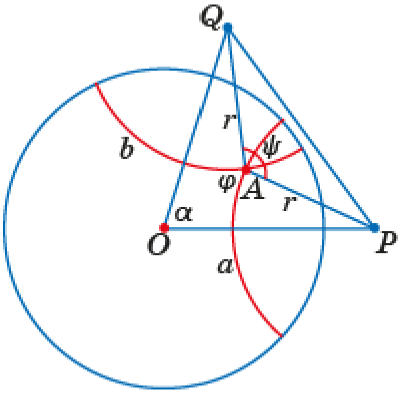


Fig. 14.17

Let Q denote the point obtained by rotating point P through angle α. Let A denote the intersection point of arcs a and b. Let ψ denote the angle PAQ. Note that ψ = 180° – φ.

By the law of cosines applied to triangle OPQ, we have: $PQ^2 = 2d^2 - 2d^2 \cos \alpha$. By the law of cosines applied to triangle APQ, we have: $PQ^2 = 2r^2 - 2r^2 \cos \psi$, where r is the radius of the circle generating arc a, with $r^2 = d^2 - 1$.

Consequently, we have the equality:

$$d^2(1 - \cos \alpha) = (d^2 - 1)(1 - \cos \psi),$$

from which we obtain the formula for the squared distance d:

$$d^2 = \frac{1 - \cos \psi}{\cos \alpha - \cos \psi} = \frac{1 + \cos \varphi}{\cos \alpha + \cos \varphi}.$$

Substituting α = 72°, φ = 90° into this formula, we find $d = \sqrt{1 + \sqrt{5}}$.

Figure 14.18 shows a fragment of a regular parquet made of equal regular pentagons with angles of 90°, obtained using the GeoGebra computer program.

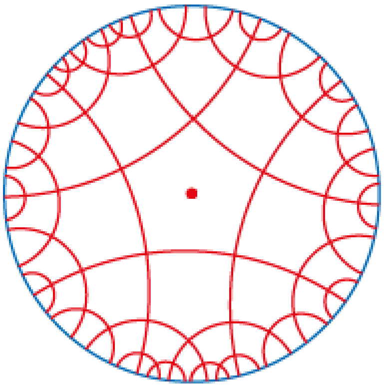


Fig. 14.18

In it, regular polygons are obtained by applying symmetries of the central regular pentagon and of the resulting regular pentagons.

Exercises

1. Obtain in the GeoGebra computer program the inversion of: a) a point; b) a line; c) a circle.

2. Construct a Lobachevsky line passing through: a) two points of the Lobachevsky plane; b) a point of the Lobachevsky plane and a point of the Poincaré circle; c) two points of the Poincaré circle.

3. Construct a Lobachevsky line perpendicular to a given Lobachevsky line and passing through a given point: a) lying on; b) not lying on the given Lobachevsky line.

4. Construct a Lobachevsky segment. Find its midpoint.

5. Construct the angle bisector of an angle formed by two intersecting Lobachevsky lines.

6. Construct a circle in the Lobachevsky plane. Find its center.

7. Construct a circle in the Lobachevsky plane with a given center, passing through a given point.

8. In the GeoGebra computer program, construct a parabola with a given directrix and the focus which is the center of the Poincaré circle.

9. In the GeoGebra computer program, construct an ellipse with a given constant c and given foci, one of which is the center of the Poincaré circle.

10. In the GeoGebra computer program, construct a hyperbola with a given constant c and given foci, one of which is the center of the Poincaré circle.

11. In the GeoGebra computer program, construct a regular triangle whose angles are equal to 45°.

12. In the GeoGebra computer program, construct a regular parquet made of regular triangles whose angles are equal to 45°.

13. In the GeoGebra computer program, construct a square whose angles are equal to 60°.

14. In the GeoGebra computer program, construct a regular parquet made of squares whose angles are equal to 60°.

15. In the GeoGebra computer program, construct a regular hexagon whose angles are equal to 90°.

16. In the GeoGebra computer program, construct a regular parquet made of regular hexagons whose angles are equal to 90°.

ANSWERS

1

3. Let point C be located in the interior region of a parabola with focus F and directrix d. Drop a perpendicular CD from it to the directrix. Denote by A its point of intersection with the parabola (Fig. A1.1). Then $CD < CA + AF = CA + AD = CD$. Hence, the distance from point C to the focus F is less than the distance to the directrix d.

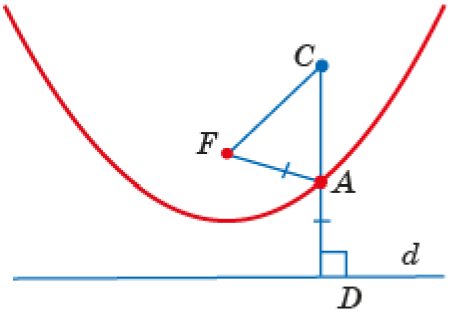


Fig. A1.1

4. a) Point A belongs to the parabola. The desired tangent is the line containing the angle bisector of $\angle FAD$ (Fig. A1.2, a); b) draw a circle with center at point B and radius BF. Denote by D_1, D_2 its points of intersection with the directrix. Draw the perpendicular bisectors a_1 and a_2 of segments FD_1 and FD_2, respectively. These will be the desired tangents to the parabola (Fig. A1.2, b).

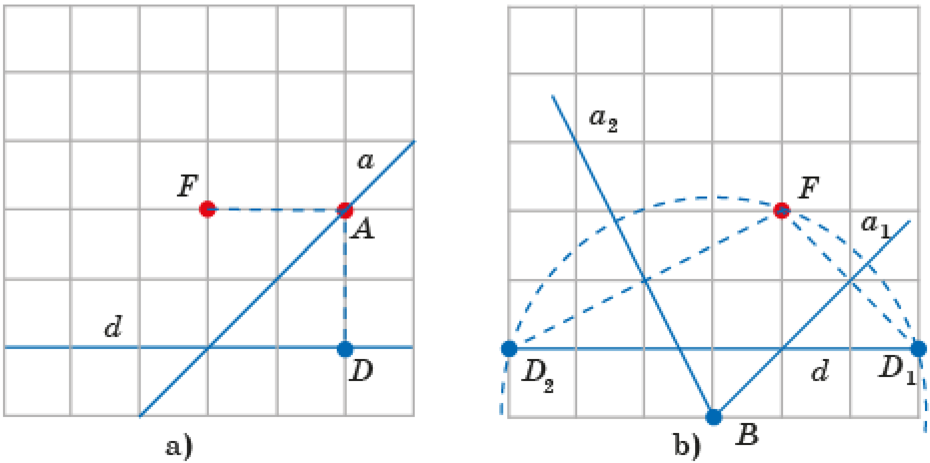


Fig. A1.2

5. a) One; b) two; c) none.

6. With centers at points A_1, A_2 and radii equal to the distances from these points to the directrix, draw circles. The desired foci will be the intersection points of these circles (Fig. A1.3). If the distance between the given points is less than the sum of the radii of the circles and greater than their difference, there are two foci. If the distance between the given points equals the sum or the difference of the radii of the circles, there is one focus. If the distance between the given points is greater than the sum of the radii of the circles or less than their difference, there are no foci.

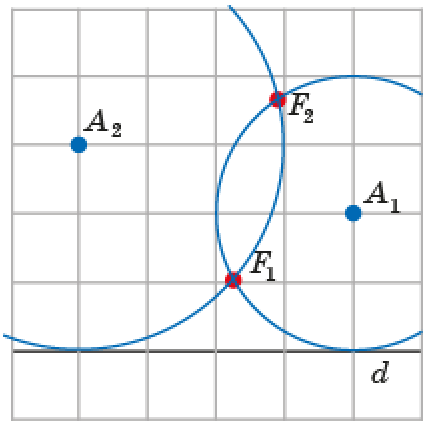


Fig. A1.3

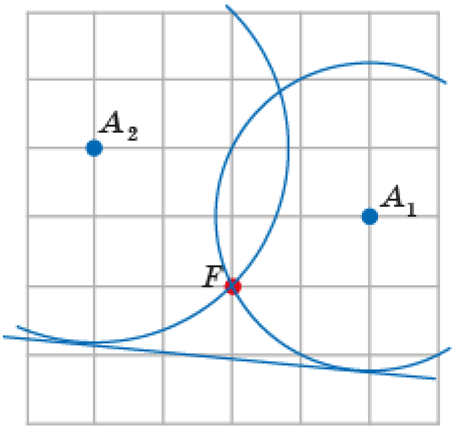


Fig. A1.4

7. With centers at points A_1, A_2 and radii equal to the distances from these points to the focus, draw circles. The desired directrix will be the common external tangent to these circles (Fig. A1.4). If the distance between the given points is less than the difference of the radii of these circles, there is no directrix. In other cases, there are two directrices. **8.** A parabola with focus *A* and directrix *b* (Fig. A1.5).

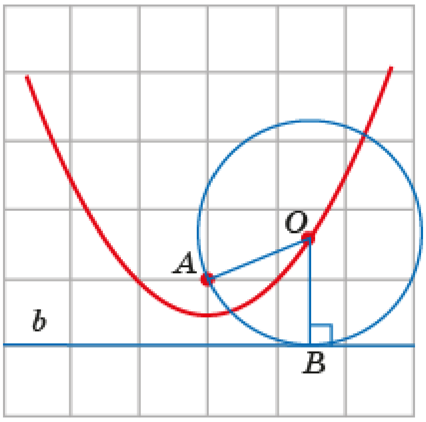


Fig. A1.5

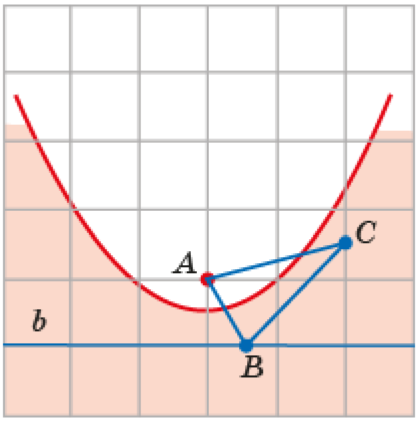


Fig. A1.6

9. Points of the parabola and points located in the exterior region (Fig. A1.6).
10. A parabola and a ray (Fig. A1.7).

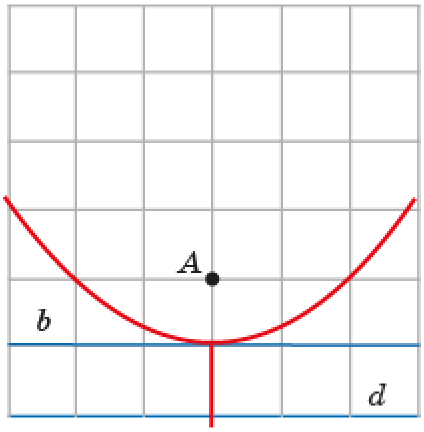


Fig. A1.7

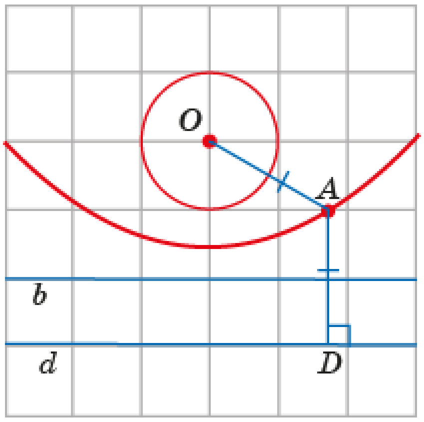
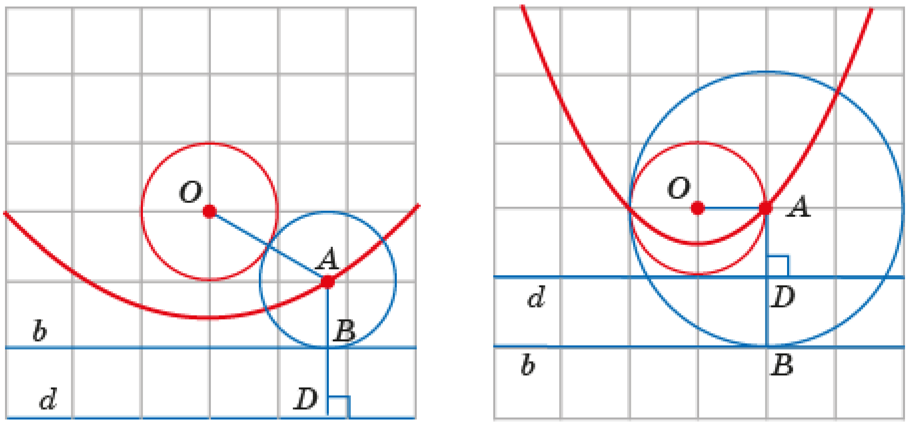


Fig. A1.8

11. A parabola with focus *O* and directrix *d* (Fig. A1.8).

12. a) A parabola with focus O and directrix d (Fig. A1.9, a); b) a parabola with focus O and directrix d (Fig. A1.9, b).



a) b)

Fig. A1.9

13. 90° (Fig. A1.10). **14.** Let us draw the bisector c of the angle formed by the given lines a and b. Construct a parabola with focus C and directrix a. Their intersection points O_1, O_2 will be the centers of the required circles.

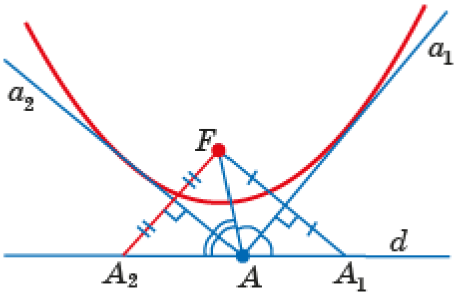


Fig. A1.10

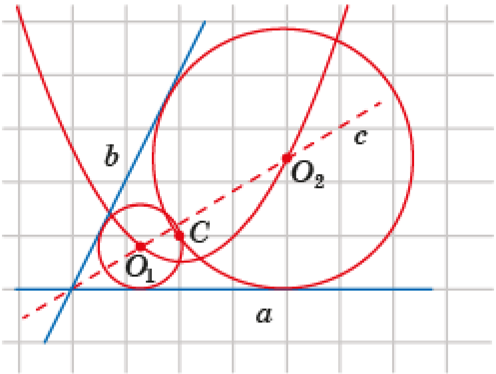


Fig. A1.11

15. From the points of tangency A_1 and A_2, we drop perpendiculars A_1D_1 and A_2D_2 to the directrix d (Fig. A1.12). The tangents a_1 and a_2 contain the bisectors of the angles FAD_1 and FAD_2, respectively. Triangles AFA_1 and AD_1A_1, AFA_2 and AD_2A_2 are congruent. Consequently, the angles AFA_1 and AD_1A_1, AFA_2 and AD_2A_2 are equal. From the equality of segments AD_1, AF, and AD_2, it follows that the angles AD_1D_2 and AD_2D_1 are equal. Therefore, the angles AFA_1 and AFA_2 are equal.

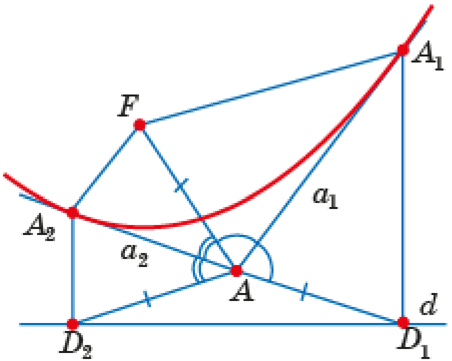


Fig. A1.12

16. a) From the previous exercise, the equality of angles B_1FB and B_1FA_1, and B_2FB and B_2FA_2 follows. Consequently, angle B_1FB_2 is half of angle A_1FA_2, which is independent of the position of point B, $\angle A_1AA_2 = 180° - \alpha$; b) with the center at point B_1 and radius B_1F, we draw a circle. Let E_1 and G_1 be its intersection points with the directrix d (Fig. A.1.13).

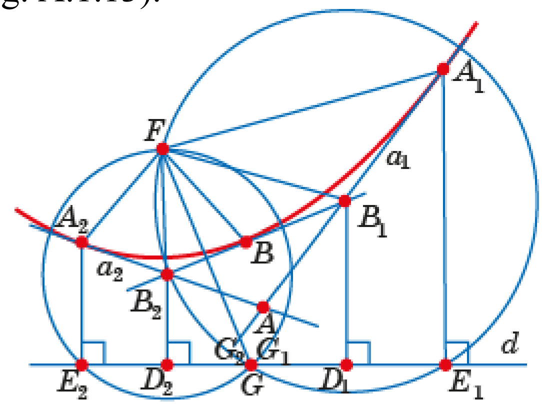


Fig. A1.13

Since the line B_1B_2 is tangent to the parabola, it is the perpendicular bisector of the segment FG_1. With the center at point B_2 and radius B_2F, we draw a circle. Let E_2 and G_2 be its intersection points with the directrix. Since the line B_1B_2 is tangent to the parabola, it is the perpendicular bisector of the segment FG_2. Since this perpendicular bisector is the same, points G_1 and G_2 coincide. Let's denote it as G. Since $GD_1 = D_1E_1$ and $GD_2 = D_2E_2$, the segment D_1D_2 is half the segment E_1E_2, which is independent of the position of point B.

2

1. The required ellipse is shown in Figure A2.1.

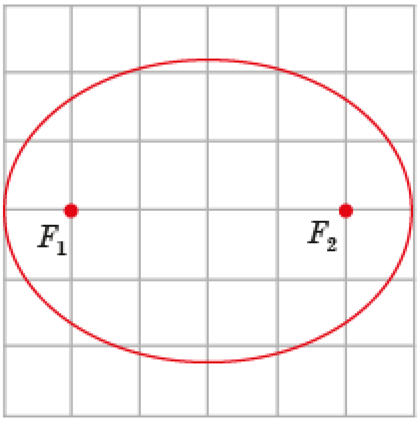


Fig. A2.1

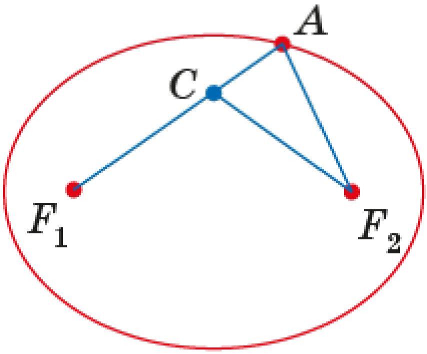


Fig. A2.2

3. Let point C lie inside the ellipse with foci F_1, F_2 and constant c. Denote by A the intersection point of ray F_1C with the ellipse (Fig. A2.2). Then $CF_1 + CF_2 < AF_1 + AF_2 = c$. Hence, the sum of the distances from point C to the foci F is less than the constant c.

4. Construct point F', symmetric to point F_2 with respect to tangent line a. Draw line F_1F'. The intersection point A of this line with the tangent is the desired point of tangency (Fig. A2.3).

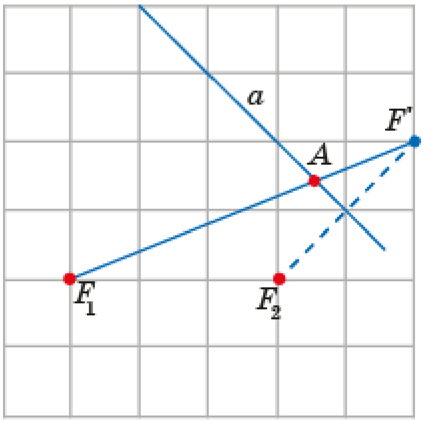


Fig. A2.3

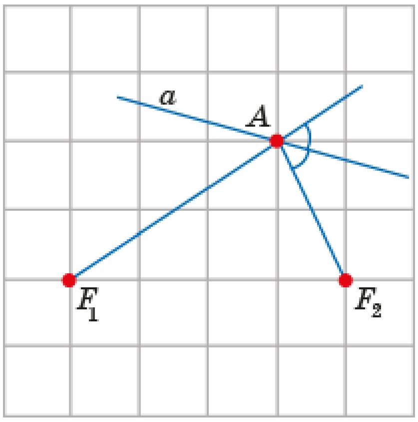


Fig. A2.4

5. Draw ray F_1A. The line a containing the bisector of the angle adjacent to angle $F_1AF_2$is the desired tangent (Fig. A2.4).

6. Draw a circle centered at point B with radius BF_2. Draw a circle centered at point F_1 with radius c. Since point Blies outside the ellipse, the inequality $BF_1 + BF_2 > c$holds. Consequently, the distance between the centers of these circles is less than the sum and greater than the difference of their radii. Hence, these circles intersect at two points. Denote them F' and F''. The desired tangents to the ellipse are the perpendicular bisectors a_1, a_2 of segments F_2F' and F_2F'', respectively (Fig. A2.5). **7.** a) One; b) two; c) none.

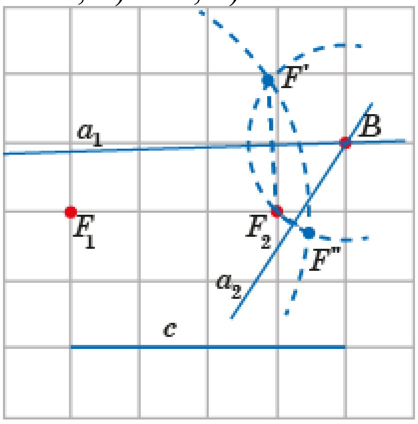


Fig. A2.5

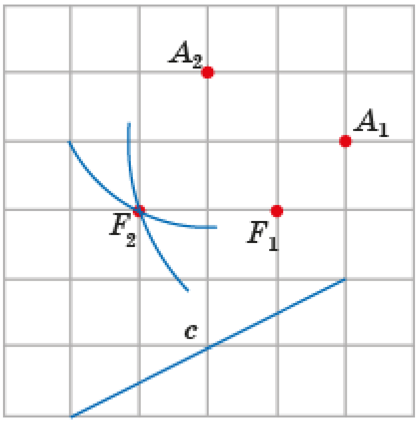


Fig. A2.6

8. With centers at points A_1, A_2 and radii $c - AF_1$, $c - CF_2$respectively, construct circles. The desired second focus F_2 is the intersection point of these circles (Fig. A2.6). **9.** Construct point F_2', symmetric to point F_2 with respect to tangent line a (Fig. A2.7). The length of segment F_1F' is the desired value of constant c.

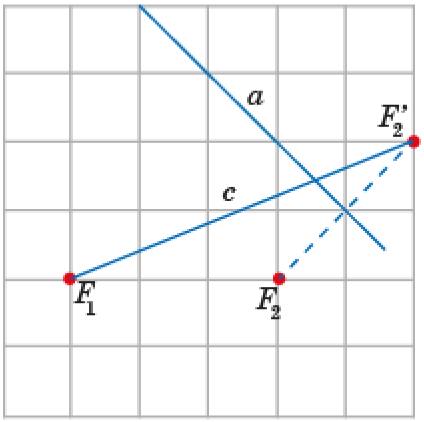


Fig. A2.7

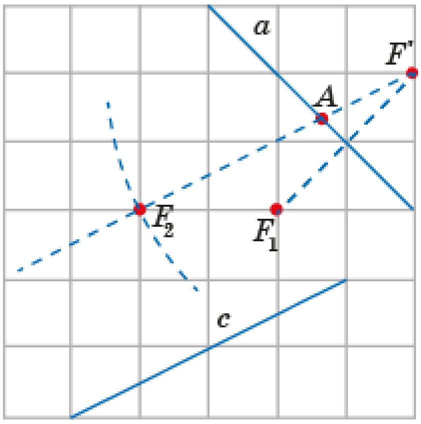


Fig. A2.8

10. Construct point F', symmetric to point F_1 with respect to tangent line a. Draw ray $F'A$. Draw a circle centered at point F' with radius c. Its intersection with ray $F'A$ is the desired second focus F_2 (Fig. A2.8).

11. Construct point F', symmetric to point F_1 with respect to tangent line b. Draw circles centered at points F' and A with radii c and $c - AF_1$, respectively. The desired second focus F_2 is the intersection point of these circles (Fig. A2.9).

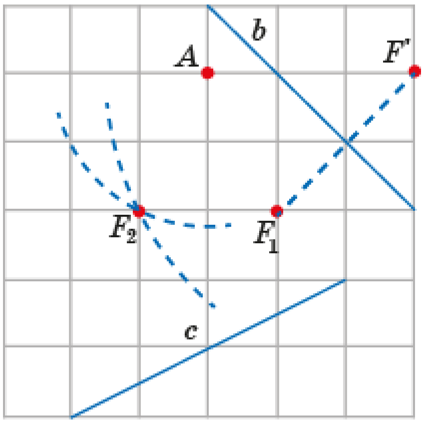


Fig. A2.9

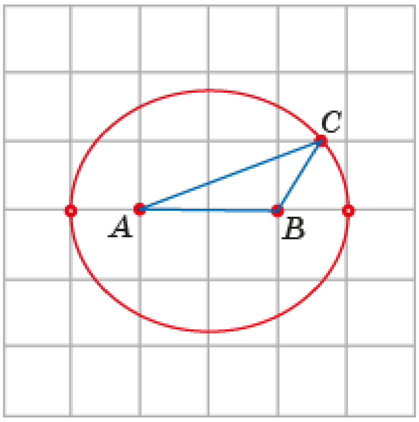


Fig. A2.10

12. The desired locus is an ellipse with foci A, B, excluding the two points lying on line AB (Fig. A2.10).

13. The desired locus is an ellipse with foci O and P (Fig. A2.11).

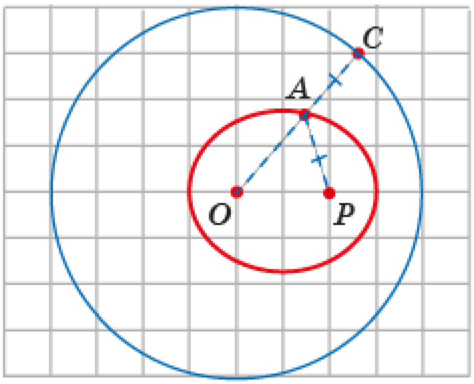


Fig. A2.11

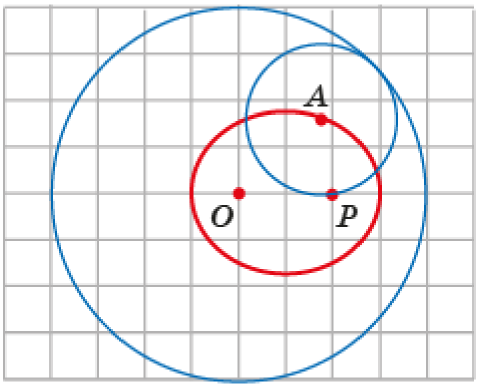


Fig. A2.12

14. The desired locus is an ellipse with foci O and P (Fig. A2.12).
15. The desired locus is an ellipse with foci $O_1$and O_2 (Fig. A2.13).

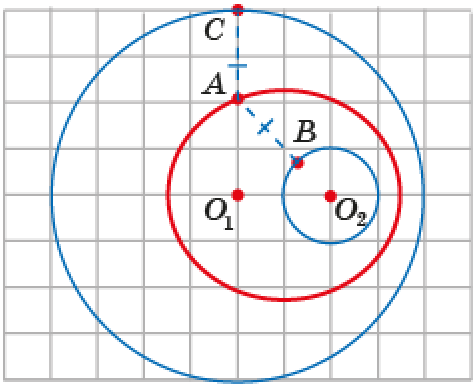


Fig. A2.13

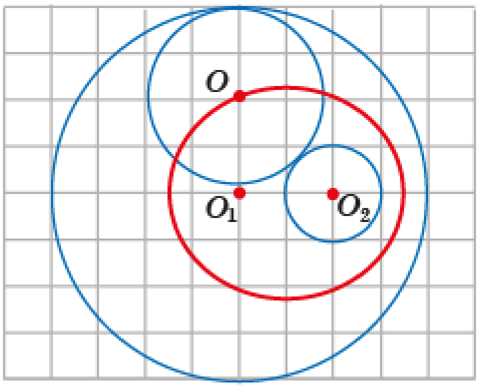


Fig. A2.14

16. The desired locus is an ellipse with foci $O_1$and O_2 (Fig. A2.14).
17. The desired locus is an ellipse with foci $O_1$and O_2 (Fig. A2.15).

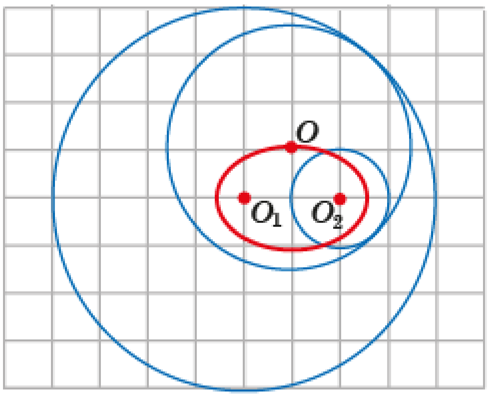


Fig. A2.15

18. Let us prove, for example, the equality of angles AF_1A_1 and AF_1A_2 (Fig. A2.16). From the construction of the tangents, the equality of triangles $AF_1F'_2$ and $AF_1F''_2$ follows. Therefore, angles AF_1A_1 and AF_1A_2 are equal.

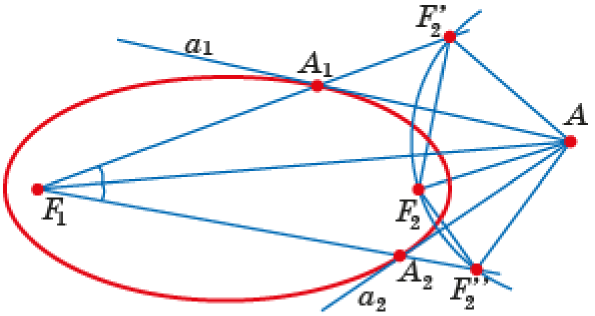


Fig. A2.16

19. Let F'_1, F'_2 be points symmetric to points F_1, F_2 with respect to the corresponding tangents a_1, a_2 (Fig. A2.17). Triangles AF'_1F_2 and $AF_1F'_2$ are congruent by three sides. Therefore, $\angle F'_1AF_2 = \angle F_1AF'_2$. This means that angles $F_1A\ F'_1$ and $F_2AF'_2$ are equal. Consequently, angles F_1AA_1 and F_2AA_2 are also equal.

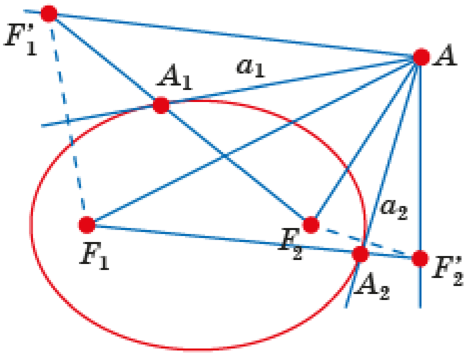


Fig. A2.17

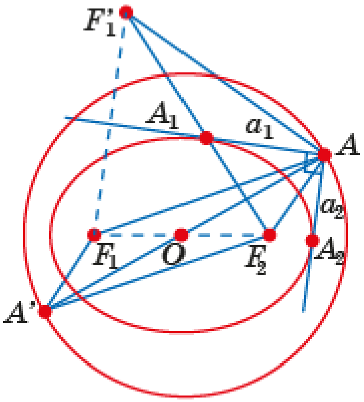


Fig. A2.18

20. Let F'_1 denote the point symmetric to point F_1 with respect to the tangent a_1 (Fig. A2.18). From the previous exercise, we obtain the equality of angles A_1AA_2 and F'_1AF_2. The angle F'_1AF_2 is equal to 90° if and only if the equality $AF_1'^2 + AF_2{}^2 = F_1'F_2{}^2$ holds. Given that $AF_1' = AF_1$ and $F_1'F_2 = c$, we obtain the equality $AF_1{}^2 + AF_2{}^2 = c^2$. Let A' denote the point symmetric to point A with respect to point O. We use the fact that the sum of the squares of the sides of a parallelogram equals the sum of the squares of its diagonals. Applying this property to parallelogram $A'F_1AF_2$, we obtain the equality $2AF_1{}^2 + 2AF_2{}^2 = 4OA^2 + d^2$, from which it follows that $4OA^2 = 2c^2 - d^2$. Thus, the angle between the tangents a_1 and a_2 is 90° if and only if $OA^2 = \frac{2c^2-d^2}{4}$. This means that point A belongs to a circle with center O and radius $R = \frac{\sqrt{2c^2-d^2}}{2}$.

21. Let us prove, for example, that the measure of angle $B_1F_1B_2$ is independent of the position of point B (Fig. A2.19). As established in the previous exercise, the following angle equalities hold: $\angle B_1F_1A_1 = \angle B_1F_1B$ and $\angle B_2F_1A_2 = \angle B_2F_1B$. Angle $B_1F_1B_2$ is equal to half the measure of angle $A_1F_1A_2$, whose magnitude is independent of the point's position. Therefore, the measure of angle $B_1F_1B_2$ is also independent of the position of point B.

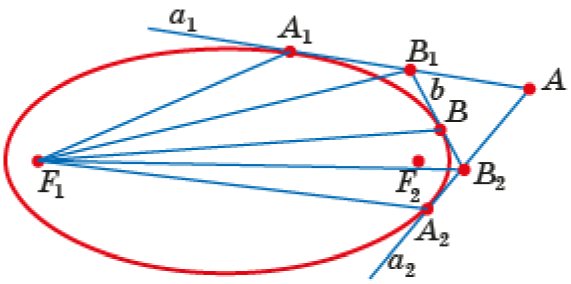


Fig. A2.19

3

1. The required hyperbola is shown in Figure A3.1.

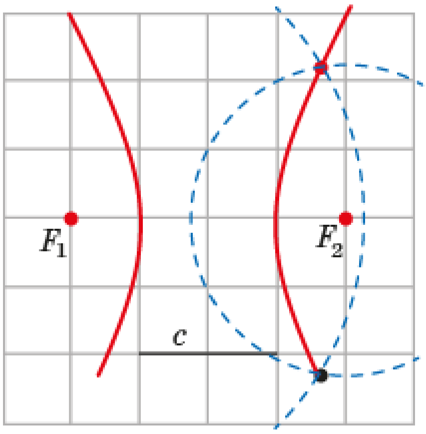


Fig. A3.1

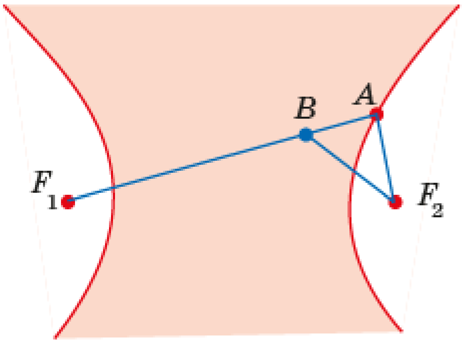


Fig. A3.2

3. Let point B lie outside the hyperbola with foci F_1, F_2 and constant c. Denote by $A$the intersection point of ray F_1B with the hyperbola (Fig. A3.2). Then $BF_1 - BF_2 = AF_1 - AB - AF_2 < c$. It can be shown similarly that $BF_2 - BF_1 < c$. Consequently, the inequality $|\ BF_1 - BF_2\ | < c$ holds.

4. Construct point F', symmetric to point F_2 with respect to line a (Fig. A3.3). The length of segment F_1F' will be the desired value c. Draw ray F_1F'. Its intersection point A with line a will be the desired point of tangency.

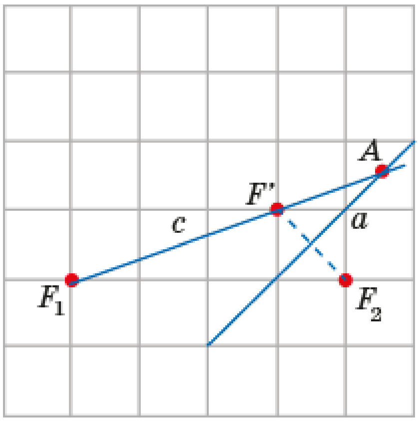


Рис. A3.3

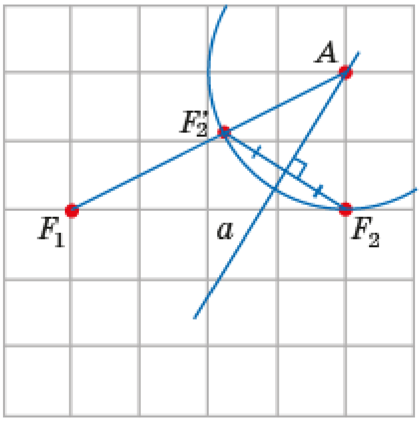


Fig. A3.4

5. Draw segment F_1A. With center at point $A$and radius AF_2, draw a circle. Denote by F_2' its intersection point with segment F_1A. The desired tangent is the perpendicular bisector of segment F_2F_2' (Fig. A3.4).

6. With center at point $B$and radius BF_2, draw a circle. With center at point $F_1$and radius c, draw a circle. Denote by F_2' the intersection point of these circles. The desired tangent is the perpendicular bisector of segment F_2F_2' (Fig. A3.5).

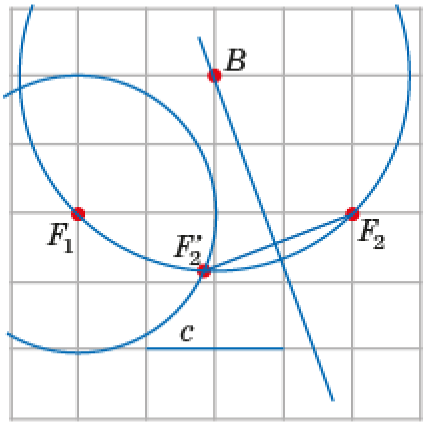


Fig. A3.5

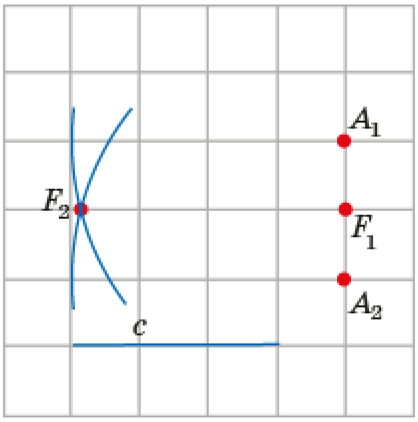


Fig. A3.6

7. a) One; b) two; c) none. **8.** Draw a circle with center at point $A_1$and radius $c + A_1F_1$. Draw a circle with center at point A_2 and radius $c + A_2F_2$. The desired second focus F_2will be the intersection point of these circles (Fig. A3.6).

9. Construct point F', symmetric to point $F_1$with respect to tangent line a. Draw ray AF'. Draw a circle with center at point $F'$and radius c. The desired second focus F_2 will be the intersection point of this circle with ray AF' (Fig. A3.7).

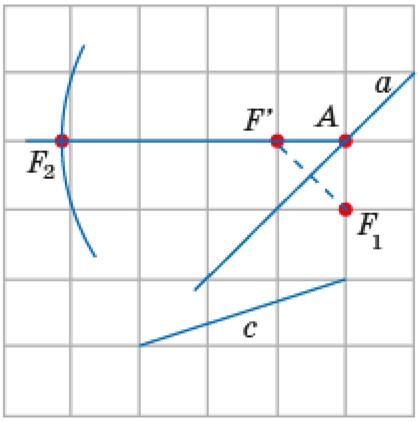


Fig. A3.7

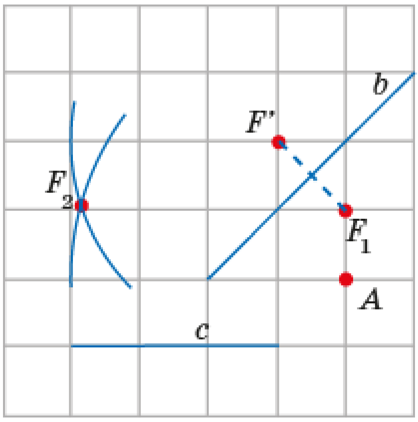


Fig. A3.8

10. Construct point F', symmetric to point F_1 with respect to tangent line b. Draw a circle with center at point $F'$and radius c. Draw a circle with center at point A and radius $c + AF_1$. The desired second focus F_2 will be the intersection point of these circles (Fig. A3.8).

11. Construct points $F'$and F'', symmetric to point $F_1$with respect to tangents a_1 and a_2, respectively. Draw circles with centers at points F' and $F''$and radius c. The desired second focus will be the intersection point of these circles (Fig. A3.9).

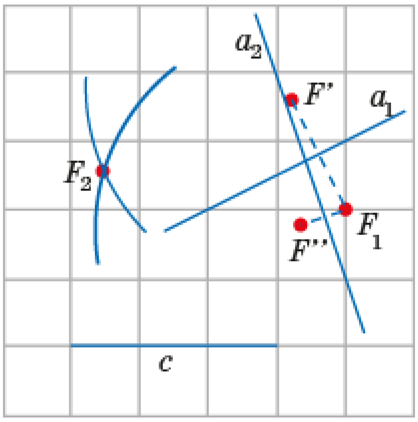


Fig. A3.9

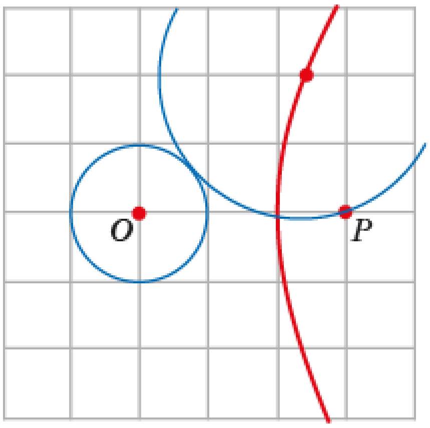


Fig. A3.10

12. The desired locus is a branch of a hyperbola whose foci are points O and P (Fig. A3.10).
13. The desired locus is a branch of a hyperbola whose foci are points O and P (Fig. A3.10).
14. The desired locus is a branch of a hyperbola whose foci are points O and P (Fig. O3.11).

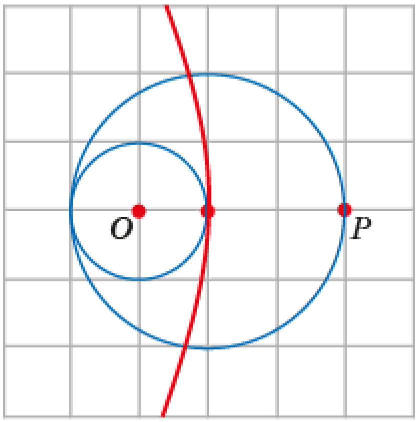


Fig. A3.11

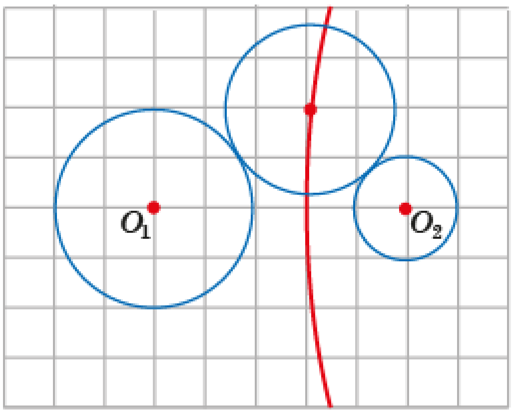


Fig. A3.12

15. The desired locus is a branch of a hyperbola whose foci are points O_1 and O_2 (Fig. A3.12). **16.** The desired locus is a branch of a hyperbola whose foci are points O_1 and O_2 (Fig. A3.13).

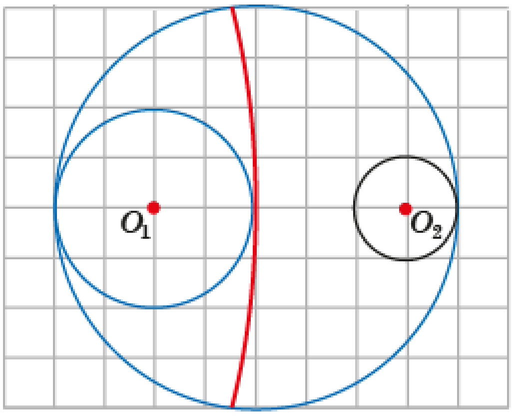


Fig. A3.13

17. Let us prove, for example, the equality of angles AF_1A_1 and AF_1A_2 (Fig. A3.14). From the construction of the tangents, the equality of triangles $AF_1F'_2$ and $AF_1F''_2$ follows. Therefore, angles AF_1A_1 and AF_1A_2 are equal.

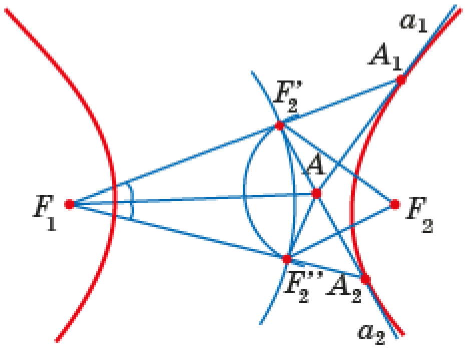


Fig. A3.14

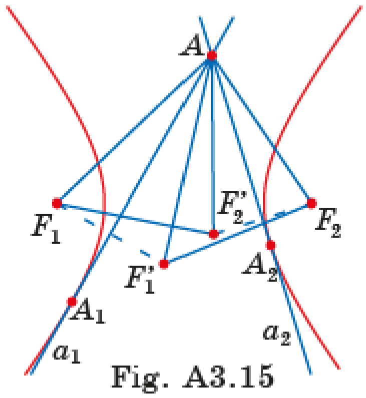


Fig. A3.15

18. Let F'_1, F'_2 be points symmetric to points F_1, F_2 with respect to the corresponding tangents a_1, a_2 (Fig. A3.15). Triangles AF'_1F_2 and $AF_1F'_2$ are congruent by three sides. Therefore, $\angle F'_1AF_2 = \angle F_1AF'_2$. This means that angles $F_1A\ F'_1$ and $F_2AF'_2$ are equal. Consequently, angles F_1AA_1 and F_2AA_2 are also equal.

19. Consider the case of the position of point A, as shown in the figure A3.16. Let F'_1 denote the point symmetric to point F_1 with respect to the tangent a_1.

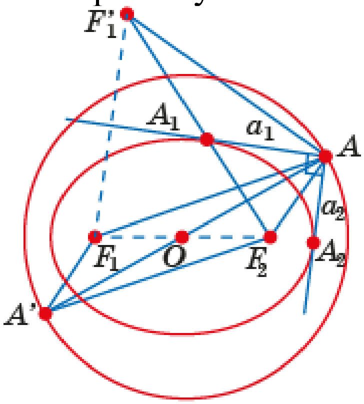


Fig. A2.18

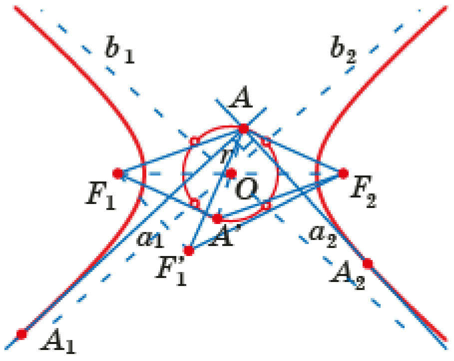


Fig. A3.16

From the previous exercise, we obtain the equality of angles A_1AA_2 and F'_1AF_2. The angle F'_1AF_2 is equal to 90° if and only if the equality $AF_1'^2 + AF_2^2 = F_1'F_2^2$ holds. Given that $AF_1' = AF_1$ and $F_1'F_2 = c$, we obtain the equality $AF_1^2 + AF_2^2 = c^2$. Let A' denote the point symmetric to point A with respect to point O. We use the fact that the sum of the squares of the sides of a parallelogram equals the sum of the squares of its diagonals. Applying this property to parallelogram $A'F_1AF_2$, we obtain the equality $2AF_1^2 + 2AF_2^2 = 4OA^2 + d^2$, from which it follows that $4OA^2 = 2c^2 - d^2$. Thus, the angle between the tangents a_1 and a_2 is 90° if and only if $OA^2 = \frac{2c^2-d^2}{4}$. This means that point A belongs to a circle with center O and radius $r = \frac{\sqrt{2c^2-d^2}}{2}$.

20. Let us prove, for example, that the measure of angle $B_1F_1B_2$ is independent of the position of point B (Fig. A3.17). As established in the previous exercise, the following angle equalities hold: $\angle B_1F_1A_1 = \angle B_1F_1B$ and $\angle B_2F_1A_2 = \angle B_2F_1B$. Angle $B_1F_1B_2$ is equal to half the measure of angle $A_1F_1A_2$, whose magnitude is independent of the point's position. Therefore, the measure of angle $B_1F_1B_2$ is also independent of the position of point B.

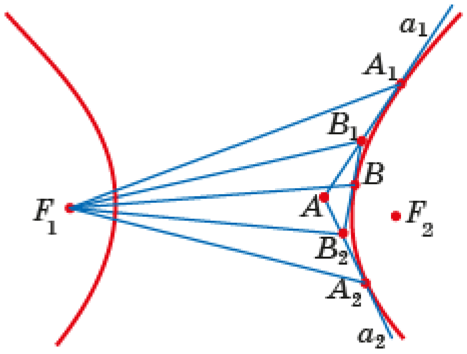


Fig. A3.17

4

1. To obtain the Conchoid of Nicomedes in the GeoGebra computer program, do the following.

Using the "Slider" tool, create a slider l varying from 1 to 3 with step 1.

Using the "Line" tool, construct a line c.

Using the "Point" tool, mark a point P at a distance of 2 from this line. Mark a point C on line c.

Construct line PC.

Using the "Circle with center and radius" tool, construct a circle with center C and radius l.

Using the "Intersect" tool, find the intersection points of this circle and line PC. Label them A and B.

In the properties of these points, select the "Show trace" option. Set $l = 1$.

If you move point C along line c, points A and B will leave a trace in the form of the Conchoid of Nicomedes corresponding to this value of l (Fig. A4.1).

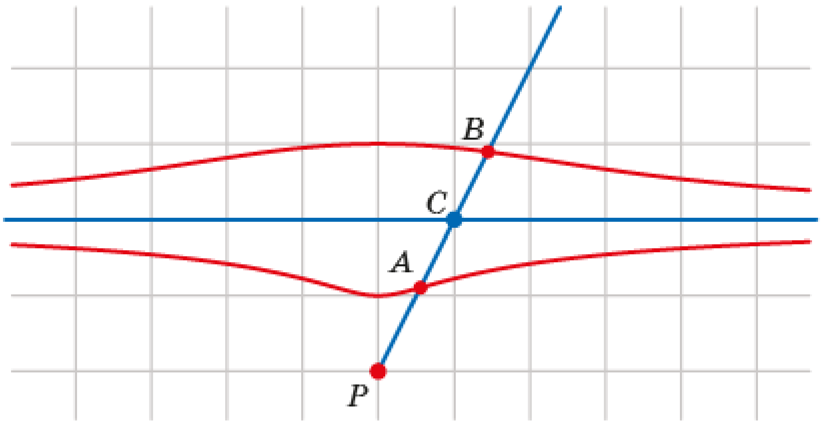


Fig. A4.1

Set $l = 2$.

If you move point C along line c, points A and B will leave a trace in the form of the Conchoid of Nicomedes corresponding to this value of l (Fig. A4.2).

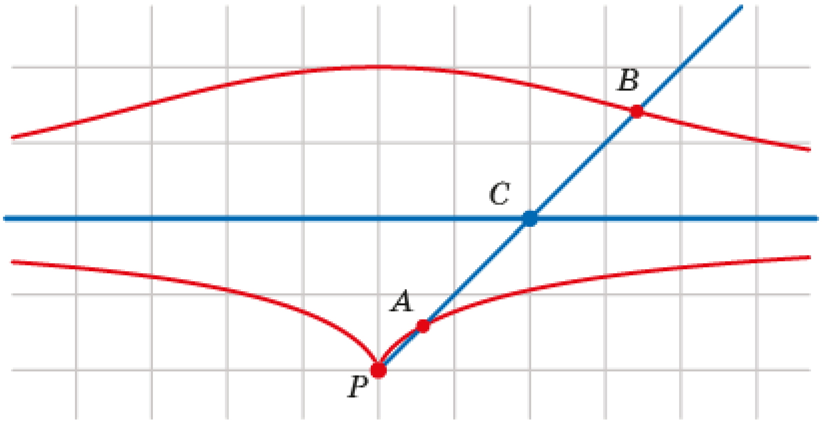


Fig. A4.2

Set $l = 3$.

If you move point C along line c, points A and B will leave a trace in the form of the Conchoid of Nicomedes corresponding to this value of l (Fig. A4.3).

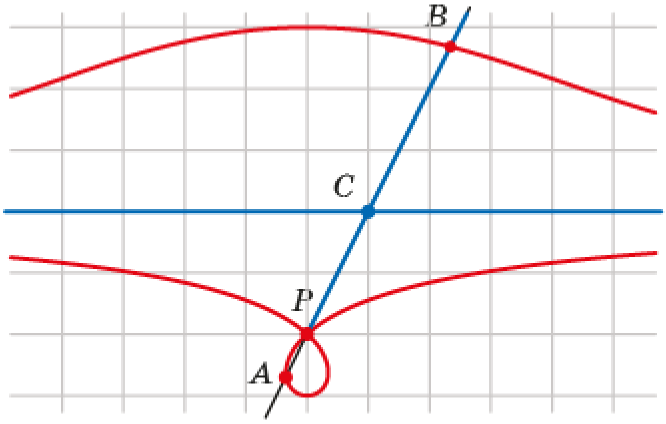


Fig. A4.3

2. To obtain the Limaçon of Pascal in the GeoGebra computer program, do the following.

Using the "Slider" tool, create a slider $l$varying from 1 to 3 with step 1.

Using the "Circle" tool, construct a circle with center O and radius 1.

Using the "Point" tool, mark points P and C on this circle.

Construct line PC.

Using the "Circle with center and radius" tool, construct a circle with center $C$and radius l.

Using the "Intersect" tool, find the intersection points of this circle and line PC. Label them A and B.

In the properties of these points, select the "Show trace" option. Set $l = 1$.

If you move point *C* along the circle, points *A* and *B* will leave a trace in the form of the Limaçon of Pascal corresponding to this value of *l* (Fig. A4.4).

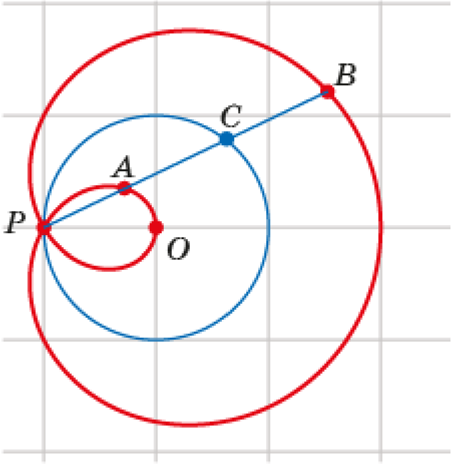


Fig. A4.4

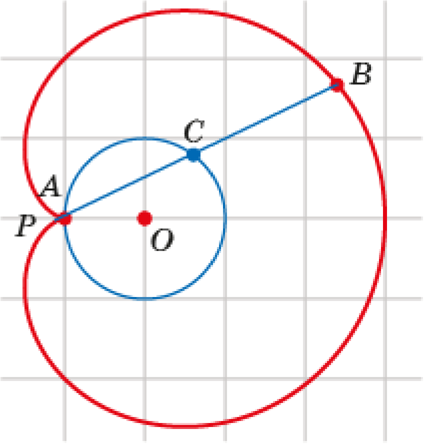


Fig. A4.5

Set $l = 2$.

If you move point *C* along the circle, points *A* and *B* will leave a trace in the form of the Limaçon of Pascal corresponding to this value of *l* (Fig. A4.5).

Set $l = 3$.

If you move point *C* along the circle, points *A* and *B* will leave a trace in the form of the Limaçon of Pascal corresponding to this value of *l* (Fig. A4.6).

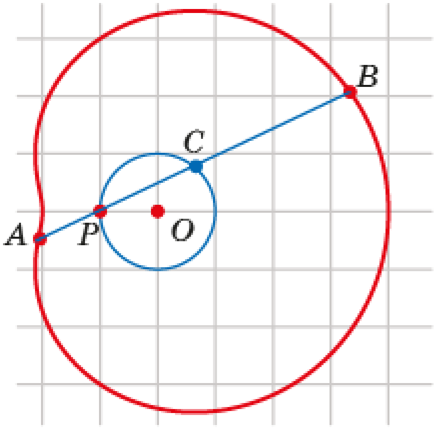


Fig. A4.6

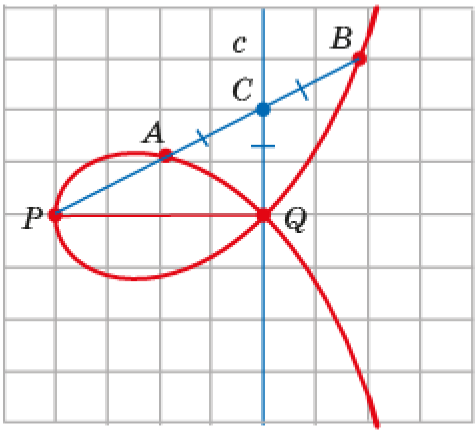


Fig. A4.7

3. To obtain the Strophoid of Nicomedes in the GeoGebra computer program, do the following.

Using the "Line" tool, construct a line *c*.

Using the "Point" tool, mark points *Q* and *C* on it.

Using the "Perpendicular line" tool, construct a line passing through point *Q* and perpendicular to line *c*.

Mark on it a point *P* located at a distance of 4 from line *c*.

Draw line *PC*.

Using the "Circle with center and radius" tool, construct a circle with center C and radius CQ.

Using the "Intersect" tool, find the intersection points of this circle and line PC. Label them A and B.

In the properties of these points, select the "Show trace" option. If you move point C along line c, points A and B will leave a trace in the form of the Strophoid of Nicomedes (Fig. A4.7).

4. To obtain the Cissoid of Diocles in the GeoGebra computer program, do the following.

Using the "Line" tool, construct a line c.

Using the "Point" tool, mark points Q and C on it.

Using the "Perpendicular line" tool, construct a line passing through point Q and perpendicular to line c.

Mark on it a point P located at a distance of 4 from line c.

Draw line PC.

Using the "Circle with center and radius" tool, construct a circle with center C and radius CQ.

Using the "Intersect" tool, find the intersection points of this circle and line PC. Label them A and B.

In the properties of these points, select the "Show trace" option. If you move point C along line c, points A and B will leave a trace in the form of the Cissoid of Diocles (Fig. A4.8).

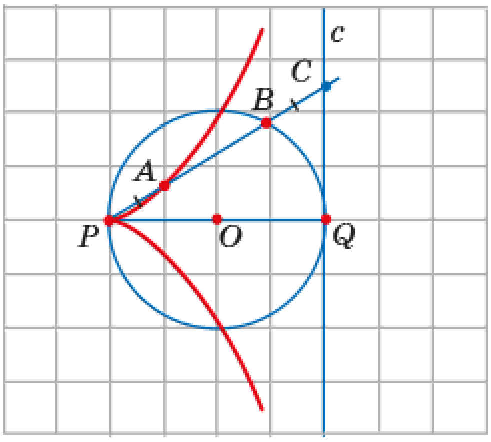


Fig. A4.8

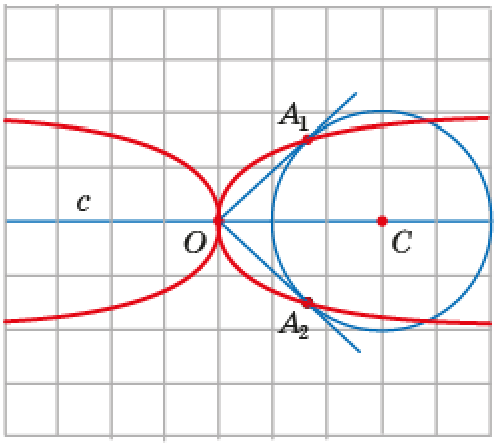


Fig. A4.9

5. To obtain the Kappa curve in the GeoGebra computer program, do the following.

Using the "Line" tool, construct a line c.

Using the "Point" tool, mark points O and C on it.

Using the "Circle with center and radius" tool, construct a circle with center C and radius 1.

Using the "Tangents" tool, construct the tangents to this circle passing through point O.

Using the "Intersect" tool, find the points of tangency. Label them A_1, A_2.

In the properties of these points, select the "Show trace" option.

If you move point C along line c, points $A_1$and A_2will leave a trace in the form of the Kappa curve (Fig. O4.9).

6. To obtain the Lemniscate of Bernoulli in the GeoGebra computer program, do the following.

Using the "Slider" tool, create a slider a varying from 0.8 to 5.

Using the "Point" tool, construct points $F_1, F_2$with the distance between them equal to 4.

Using the "Circle with center and radius" tool, construct a circle with center $F_1$and radius a.

Using the "Circle with center and radius" tool, construct a circle with center F_2 and radius $4/a$.

Using the "Intersect" tool, find the intersection points of these circles. Label them A_1, A_2.

In the properties of these points, select the "Show trace" option. When changing the slider value, points A_1, A_2will leave a trace in the form of the Lemniscate of Bernoulli (Fig. A4.10).

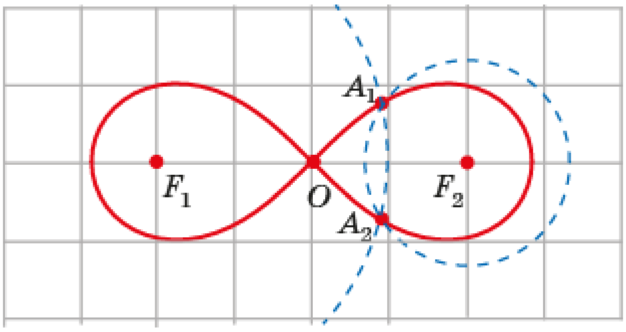


Fig. A4.10

7. To obtain the Circle of Apollonius in the GeoGebra computer program, do the following.

Using the "Point" tool, construct points A, B with the distance between them equal to 3.

Using the "Slider" tool, create a slider a varying from 2 to 6.

Using the "Circle with center and radius" tool, construct a circle with center A and radius a.

Using the "Circle with center and radius" tool, construct a circle with center B and radius $a/2$.

Using the "Intersect" tool, find the intersection points of these circles. Label them C_1, C_2.

In the properties of these points, select the "Show trace" option. When changing the slider value, points C_1, C_2 will leave a trace in the form of the Circle of Apollonius (Fig. A4.11).

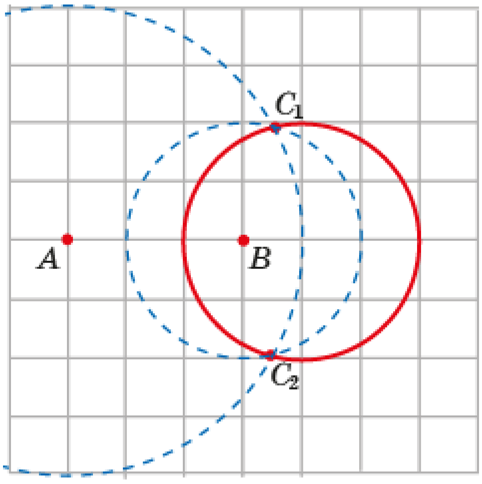


Fig. A4.11

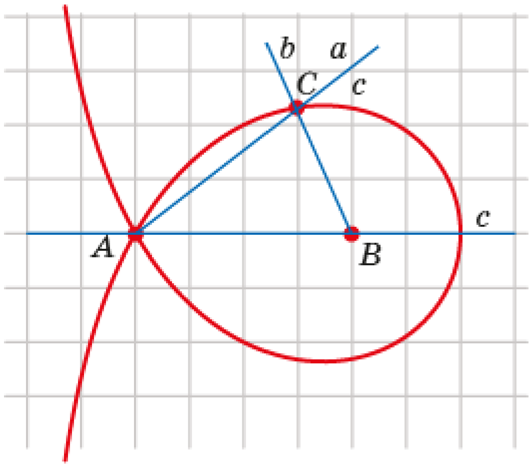


Fig. A4.12

8. To obtain the Maclaurin Trisectrix in the GeoGebra computer program, do the following.

Using the "Point" tool, construct points A, B with the distance between them equal to 4.

Using the "Line" tool, draw line AB.

Using the "Slider" tool, create a slider $t$varying from 0 to 2π.

Using the "Rotate around point" tool, rotate line ABaround point $A$by angle $t$counterclockwise.

Using the "Rotate around point" tool, rotate line ABaround point $B$by angle $3t$ counterclockwise.

Using the "Intersect" tool, find the intersection point of the obtained lines. Label it C.

In the properties of this point, select the "Show trace" option. When changing the slider value, point Cwill leave a trace in the form of the Maclaurin Trisectrix (Fig. A4.12).

9. To obtain the Witch of Agnesi in the GeoGebra computer program, do the following.

Using the "Circle with center and radius" tool, construct a circle with center O and radius 1.

Draw its diameter PQ.

Mark a point A on it.

Using the "Tangents" tool, draw tangents $p$and $q$to the circle through points P and Q.

Using the "Parallel line" tool, draw a line $a$through point Aparallel to lines p and q.

Using the "Intersect" tool, find the intersection points of this line with the circle. Label them B_1, B_2.

Draw lines PB_1, PB_2.

Find their intersection points with line q. Label them C_1, C_2.

Using the "Perpendicular line" tool, draw through these points lines perpendicular to line q.

Find their intersection points with line a. Label them A_1, A_2.

In the properties of these points, select the "Show trace" option. If you move point *A* along diameter *PQ*, points A_1, A_2 will leave a trace in the form of the Witch of Agnesi (Fig. A4.13).

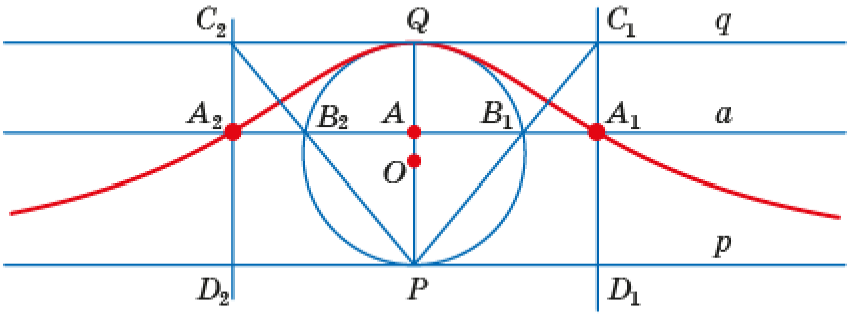


Fig. A4.13

10. Figure A4.14.

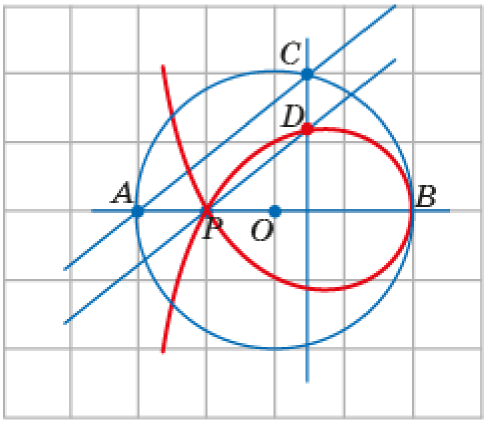


Fig. A4.14

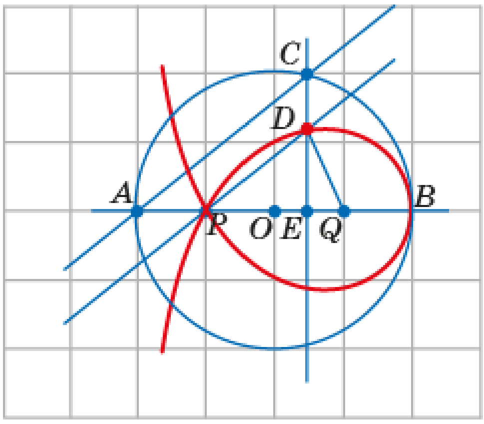


Fig. A4.15

11. Let $AB = 4$ (Fig. A4.15). Let's denote $\alpha = \angle BAC = \angle BPD$, Q – the midpoint of segment *BO*, and $\delta = PDQ$. We have $AC = 4\cos\alpha$, $AE = 4\cos^2\alpha$, $PE = 4\cos^2\alpha - 1$, $PD = \frac{4\cos^2\alpha - 1}{\cos\alpha}$. Applying the sine rule to triangle *PQD*, we get $\frac{2}{\sin\delta} = \frac{4\cos^2\alpha - 1}{\cos\alpha\sin(\alpha+\delta)}$. We can rewrite this equality as $2\cos\alpha\sin(\alpha+\delta) = \sin\delta\,(4\cos^2\alpha - 1)$. This simplifies to $\operatorname{tg}\delta = \operatorname{tg}2\alpha$. Therefore, $\delta = 2\alpha$ and $\angle BQD = 3\alpha$. This means the curve is a Maclaurin trisectrix.

12. Let the angle *BAC* be α (Fig. A4.16). Then the angle *BOC* is 2α. Lines *b* and *c* contain the altitudes of triangle *OBC*. Therefore, line *BD* is perpendicular to line *OC*. Let *Q* be the point of intersection of line *BD* and line *d*, which passes through point *O* and is perpendicular to line *AB*. The angle *OBD* is 90° – 2α. Then the angle *ODQ* is 90° – α. The angle *DOQ* is also 90° – α. Consequently, segments *QO* and *QD* are equal. Thus, point *D* belongs to the strophoid.

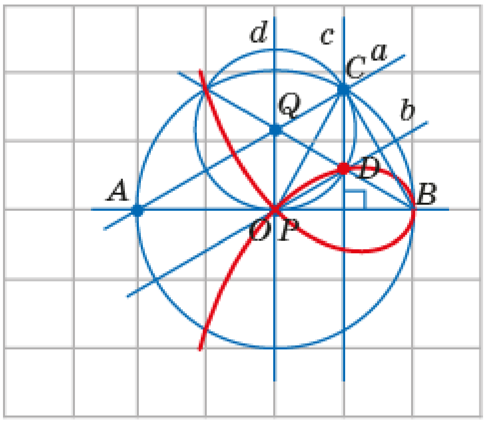


Fig. A4.16

13. Figure A4.17 shows the tangents b_1, ..., b_6, the points of tangency B_1, ..., B_6, the feet of the perpendiculars C_1, ..., C_6 dropped onto these tangents, and the curve passing through these feet.

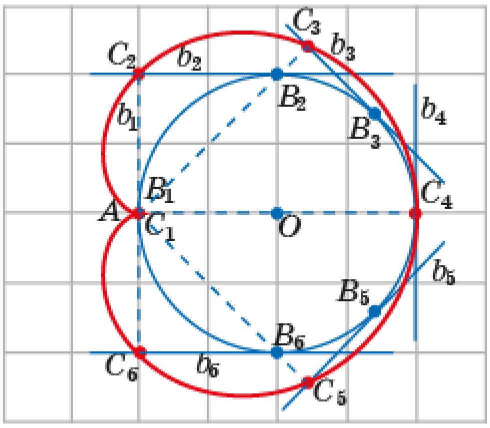


Fig. A4.17

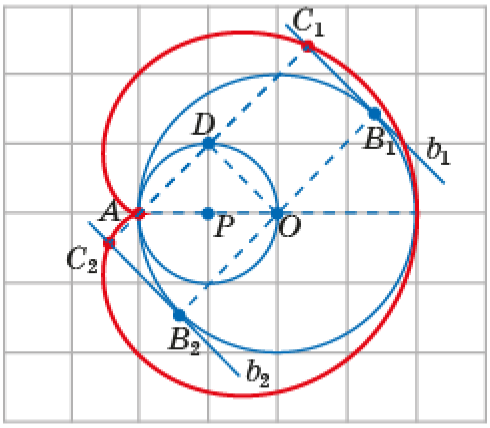


Fig. A4.18

Prove that this curve is a Pascal snail. Let's consider a circle with center O and radius l (Fig. A4.18). Draw tangents b_1, b_2 to it at diametrically opposite points B_1, B_2. From point A, drop perpendiculars AC_1, AC_2 to these tangents. Let D be the intersection point of line C_1C_2 with the circle having diameter $AO = l$. The quadrilaterals OB_1C_1D and OB_2C_2D are rectangles. Consequently, $DC_1 = DC_2 = l$. Hence, points C_1, C_2 lie on the limaçon of Pascal.

14. Figure A4.19 shows the tangents b_1, ..., b_6, the points of tangency B_1, ..., B_6, the feet of the perpendiculars C_1, ..., C_6 dropped onto these tangents, and the curve passing through these feet.

Prove that this curve is a Pascal snail. Let's consider a circle with center O and radius l (Fig. A4.20). Draw tangents b_1, b_2 to it at diametrically opposite points B_1, B_2. From point A, drop perpendiculars AC_1, AC_2 to these tangents. Let D be the intersection point of line C_1C_2 with the circle having diameter $AO = r < l$. The quadrilaterals OB_1C_1D and OB_2C_2D are rectangles. Consequently, $DC_1 = DC_2 = l$. Hence, points C_1, C_2 lie on the limaçon of Pascal.

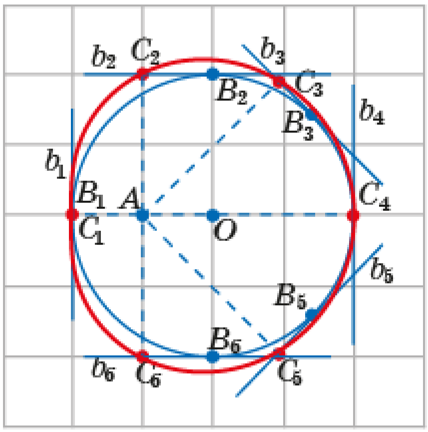


Fig. A4.19

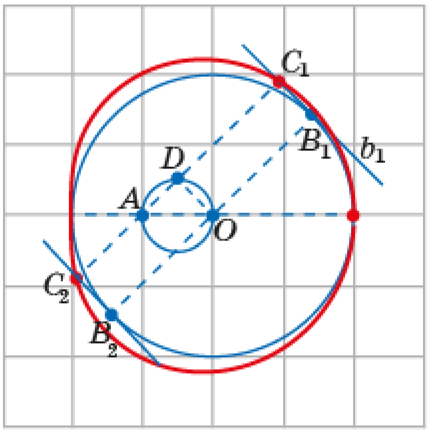


Fig. A4.20

15. Through point *B*, we draw a line *c* perpendicular to the axis *e* of the parabola (Fig. A4.21). Let *E* be the vertex of the parabola. Through it, we draw lines parallel to the tangent and the directrix. Let *D* and *M* be their intersection points with line *c* and the tangent, respectively. Through point *D*, we draw a line parallel to line *AB*. Let *C* be its intersection point with the axis of the parabola. Through point *A*, we draw a line parallel to line *c*. Let *K* be its intersection point with line *CD*. Segments *EM*, *DB*, and *KA* are equal. Triangles *FEM* and *CAK* are congruent. Therefore, the segment *CE* is independent of the choice of tangent to the parabola. We construct a circle with diameter *CE*. Point *D* belongs to this circle, and point *B* belongs to the Cramer curve. Consequently, the sought pedal curve is the Cramer curve.

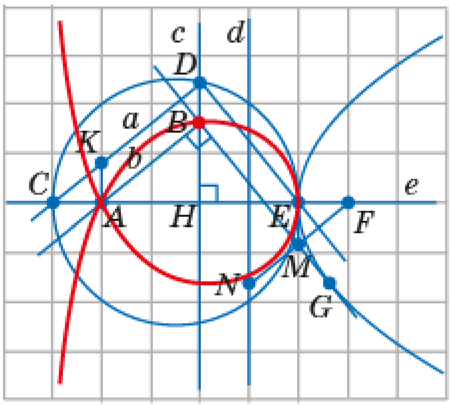


Fig. A4.21

16. Figure A4.22 shows the tangents, the points of tangency, the feet of the perpendiculars dropped onto these tangents, and the curve passing through these feet.

Let us prove that this curve is the cissoid of Diocles. Let point C_1 belong to the pedal curve of the parabola (Fig. A4.23). Recall that the tangent to the parabola passing through point *B* is the perpendicular bisector *b* of segment *FH*. The quadrilateral *AGDQ* is a rectangle. From the equality of triangles *AGC* and *DQE*, we obtain the equality of segments *AC* and *DE*. Hence, point *C* belongs to the cissoid of Diocles.

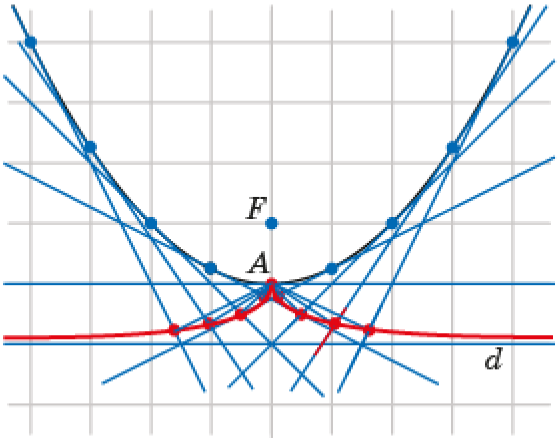


Fig. A4.22

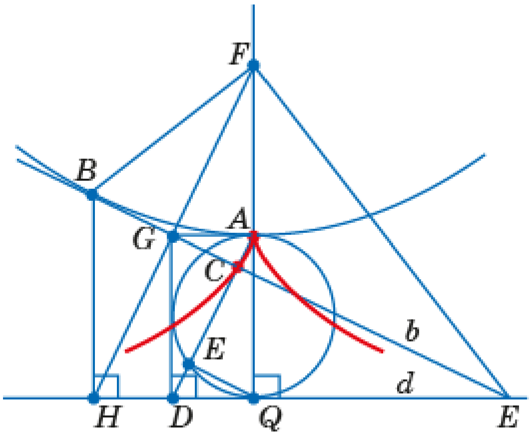


Fig. A4.23

17. Figure A4.24 shows the tangents, the points of tangency, the feet of the perpendiculars dropped onto these tangents, and the curve passing through these feet.

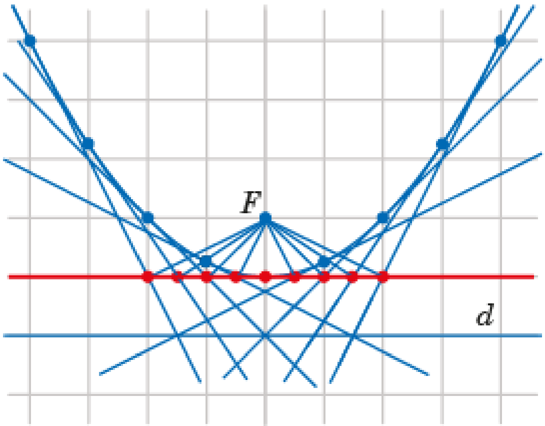


Fig. A4.24

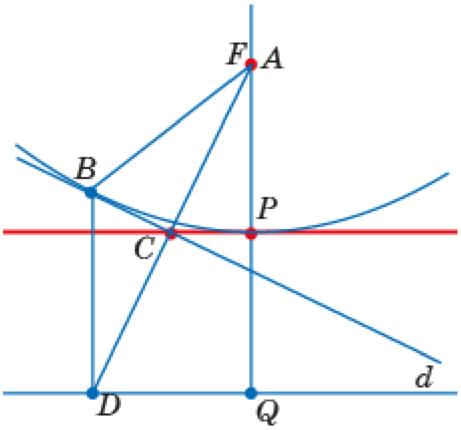


Fig. A4.25

Let us prove that this curve is the straight line. From the point of tangency B, drop a perpendicular BD to the directrix d of the parabola (Fig. A4.25). Since the perpendicular bisector of segment FD is the tangent to the parabola, the point C of the pedal curve is the midpoint of segment FD. The locus of such midpoints is a line parallel to the directrix of the parabola and passing through its vertex P.

18. With center at point F_2 and radius c, draw a circle. Choose a point A on the ellipse. Draw the ray F_2A. Denote by C its intersection point with the constructed circle. Draw segment F_1C. Through point A draw the tangent to the ellipse (Fig. A4.26). Denote by B its intersection point with segment F_1C. Triangle F_1AC is isosceles ($AF_1 = AC$). Segment AB is the angle bisector, hence also the median and altitude. Therefore, point B lies on the required pedal curve. Let D be the midpoint of segment F_2C, and O the midpoint of segment F_1F_2. Quadrilateral $OBDF_2$ is a parallelogram. Consequently, $OB = F_2D = \frac{c}{2}$. Thus, the required pedal curve is a circle with center O and radius $\frac{c}{2}$.

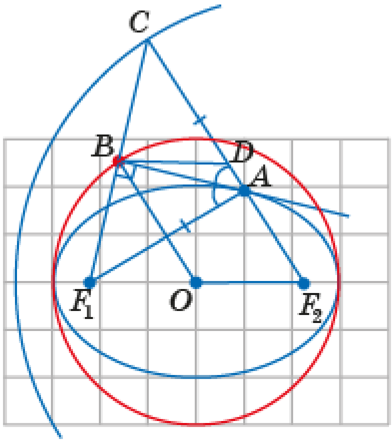


Fig. A4.26

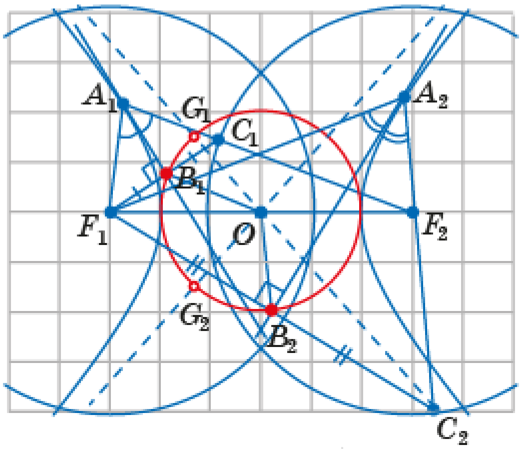


Fig. A4.27

19. Choose a point A_1 on the hyperbola. With center at point F_2 and radius c, draw a circle. Draw the ray F_2A_1. Denote by C_1 its intersection point with the constructed circle. Draw line F_1C_1. Through point A_1 draw the tangent to the hyperbola (Fig. A4.27). Denote by B_1 its intersection point with line F_1C_1. Triangle $F_1A_1C_1$ is isosceles ($A_1F_1 = A_1C_1$). Segment A_1B_1 is the angle bisector, hence also the median and altitude. Point B_1 is the foot of the perpendicular dropped from point F_1 onto the tangent to the hyperbola drawn through point A_1. Consequently, it belongs to the pedal curve. Let O be the midpoint of segment F_1F_2. Segment OB_1 is the midline of triangle $F_1F_2C_1$. Therefore, $OB_1 = \frac{F_2C}{2} = \frac{c}{2}$. Thus, the point B_1 lies on the of circle with center O and radius $\frac{c}{2}$. It is similary proved that the foot B_2 of the perpendicular dropped from point F_1 onto the tangent to the hyperbola drawn through point A_2 belongs to this circle. When the position of point A_1 changes, point B_1 describes an arc $\overset{\frown}{G_1G_2}$of a circle without its endpoints, where lines OG_1 and OG_2 are the limiting positions of the tangent. When the position of point A_2 changes, point B_2 describes an additional arc $\overset{\frown}{G_1G_2}$ of a circle without its endpoints. Consequently, the desired pedal curve is a circle with center O and radius $\frac{c}{2}$ without points G_1 and G_2.

5

2. a) Yes; b) No. **3.** $1-\frac{\sqrt{2}}{2}$, 1, $1+\frac{\sqrt{2}}{2}$, 2, $1+\frac{\sqrt{2}}{2}$, 1, $1-\frac{\sqrt{2}}{2}$. **6.** The curtate cycloid is shown in Figure A5.1.

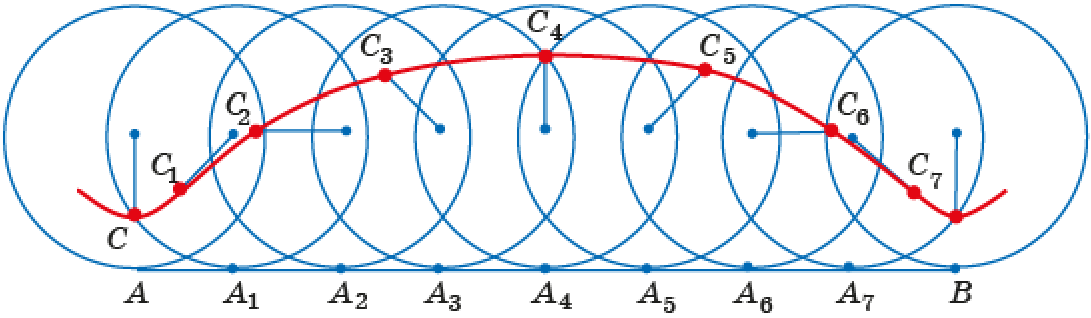


Fig. A5.1

7. To obtain a curtate cycloid in the GeoGebra software, do the following.

Create sliders for t ($-2 \le t \le 8$) and d ($0 < d < 1$).
Set, for example, $d = 0.5$.
In the "Input" bar, type $y = 0$. A line will appear.
Mark points $O(t, 1)$ and $C(t, 1-d)$.
Construct a circle with center at point O and radius 1.
Rotate point C around point O clockwise by an angle of t radians.
Connect the resulting point to point O with a segment.
In the properties of this point, select the "Show trace" option.

When the value of slider t is changed, the point will leave a trace in the form of a curtate cycloid (Fig. A5.2). If you select the "Animate" option in the slider's properties, you will obtain the desired trajectory in motion.

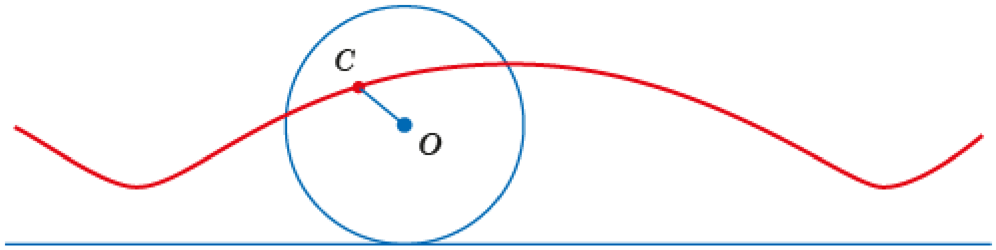


Fig. A5.2

8. The prolate cycloid is shown in Figure A5.3.

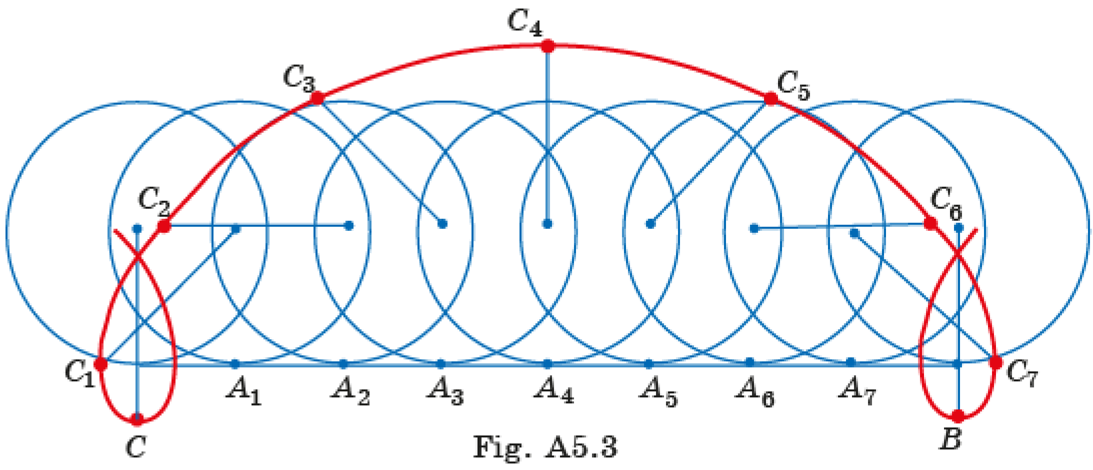


Fig. A5.3

9. To obtain a prolate cycloid in the GeoGebra software, do the following.

Create sliders for t ($-2 \le t \le 8$) and d ($1 < d < 2$).
Set, for example, $d = 1.5$.
In the "Input" bar, type y = 0. A line will appear.
Mark points $O(t, 1)$ and $C(t, 1-d)$.
Construct a circle with center at point O and radius 1.
Rotate point C around point O clockwise by an angle of t radians.
Connect the resulting point to point O with a segment.
In the properties of this point, select the "Show trace" option.

When the value of slider *t* is changed, the point will leave a trace in the form of a prolate cycloid (Fig. A5.4). If you select the "Animate" option in the slider's properties, you will obtain the desired trajectory in motion.

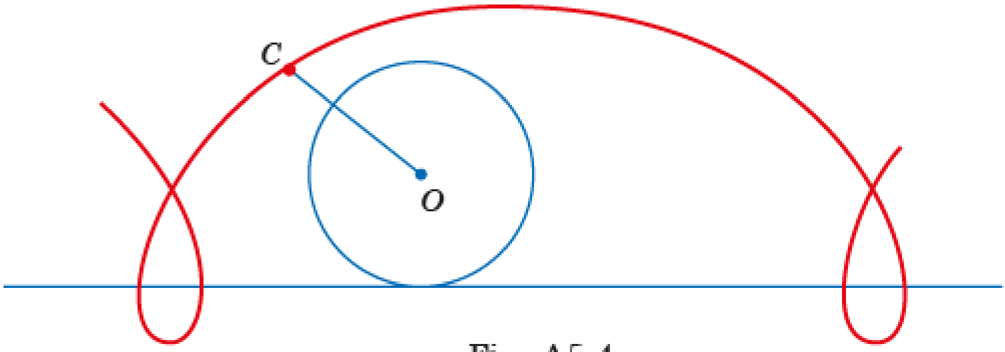


Fig. A5.4

12. The trajectory of a square's vertex is shown in Figure A5.5.

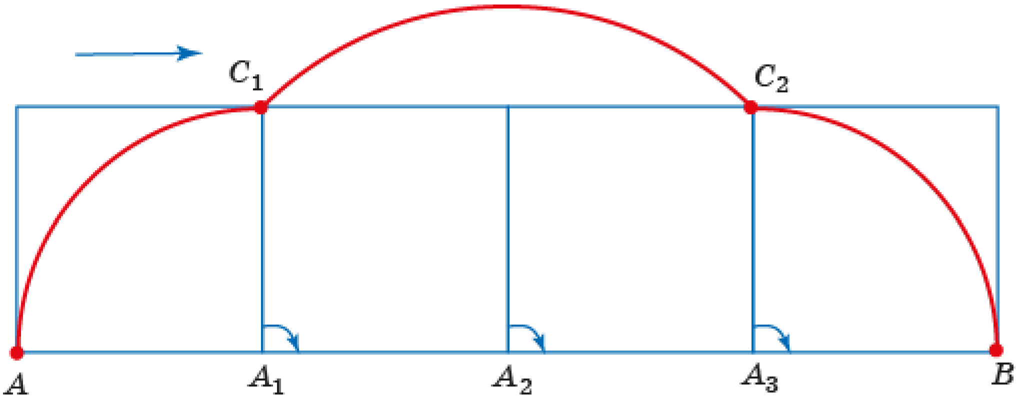


Fig. A5.5

14. To obtain the trajectory of a square's vertex in the GeoGebra software, do the following.

Create a slider for *t* (0° ≤ *t* ≤ 360°).

In the "Input" bar, type: y = 0. A line will appear.

Mark points *A*(0, 0), A_1(2, 0), A_2(4, 0), A_3(6, 0), *B*(8, 0), *C*(2, 2), *D*(6, 2).

Construct squares with vertices *A* and A_1, A_1 and A_2, A_2 and A_3, A_3 and *B*. In the "Input" bar, type:

```
If(0° ≤ t ≤ 90°, Rotate(Polygon(A, A_1, 4), -t, A_1))
If(90° ≤ t ≤ 180°, Rotate(Polygon(A_1, A_2, 4), -t + 90°, A_2))
If(180° ≤ t ≤ 240°, Rotate(Polygon(A_3, B, 4), -t + 180°, A_2))
If(0° ≤ t ≤ 90°, Rotate(A, -t, A_1))
If(90° ≤ t ≤ 180°, Rotate(C, -t + 90°, A_2))
If(80° ≤ t ≤ 240°, Rotate(D, -t +180°, A_3))
```

Designate the rotated points as C_1, C_2 accordingly.

In the properties of these points, select the "Show trace" option. When the value of the slider is changed, the point will leave a trace in the form of a square's vertex trajectory (Fig. A5.6). If you select the "Animate" option in the slider's properties, you will obtain this trajectory in motion.

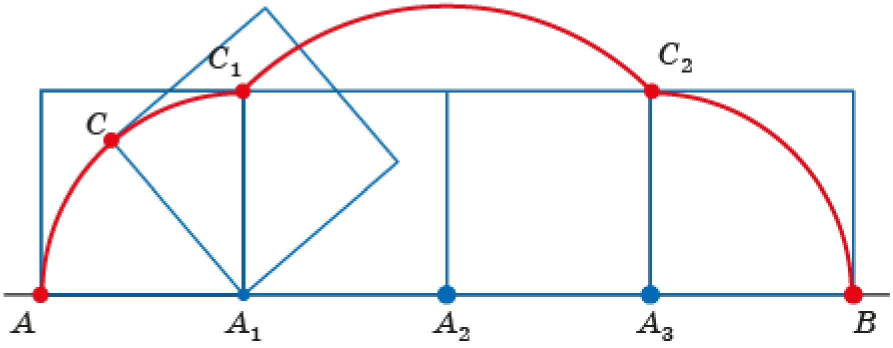


Fig. A5.6

15. The trajectory of a regular hexagon's vertex is shown in Figure A5.7.

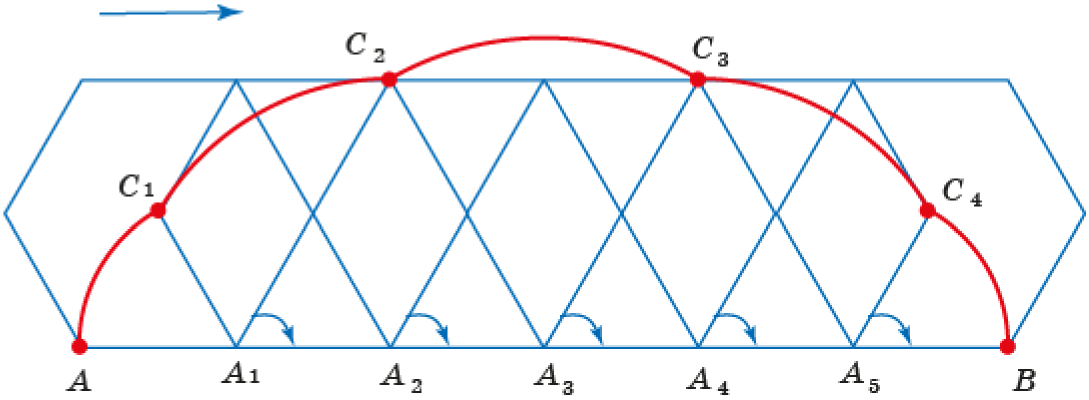


Fig. A5.7

17. To obtain the trajectory of a regular hexagon's vertex in the GeoGebra software, do the following.

Create a slider for *t* (0° ≤ t ≤ 360°).

In the "Input" bar, type: y = 0. A line will appear.

Mark points A(0, 0), A_1(2, 0), A_2(4, 0), A_3(6, 0), A_4(8, 0), A_5(10, 0), B(12, 0), C(1, $\sqrt{3}$), D(4, 2$\sqrt{3}$), E(8, 2$\sqrt{3}$), F(10, $\sqrt{3}$).

Construct regular hexagons with vertices A and A_1, A_1 and A_2, A_2 and A_3, A_3 and A_4, A_4 and A_5, A_5 and B.

In the "Input" bar, type:

If(0° ≤ t ≤ 60°, Rotate(Polygon(A, A_1, 6), -t, A_1))
If(60° ≤ t ≤ 120°, Rotate(Polygon(A_1, A_2, 6), -t + 60°, A_2))
If(120° ≤ t ≤ 180°, Rotate(Polygon(A_2, A_3, 6), -t + 120°, A_3))
If(180° ≤ t ≤ 240°, Rotate(Polygon(A_3, A_4, 6), -t + 180°, A_4))
If(240° ≤ t ≤ 360°, Rotate(Polygon(A_4, A_5, 6), -t + 240°, A_5))
If(0° ≤ t ≤ 60°, Rotate(A, -t, A_1))
If(60° ≤ t ≤ 120°, Rotate(C, -t + 60°, A_2))
If(80° ≤ t ≤ 240°, Rotate(D, -t + 120°, A_3))
If(180° ≤ t ≤ 240°, Rotate(E, -t + 180°, A_4))
If(240° ≤ t ≤ 360°, Rotate(F, -t + 240°, A_5))

Designate the rotated points as C_1, C_2, C_3, C_4 accordingly. In the properties of these points, select the "Show trace" option.

When the value of the slider is changed, the point will leave a trace in the form of a regular hexagon's vertex trajectory (Fig. A5.8). If you select the "Animate" option in the slider's properties, you will obtain this trajectory in motion.

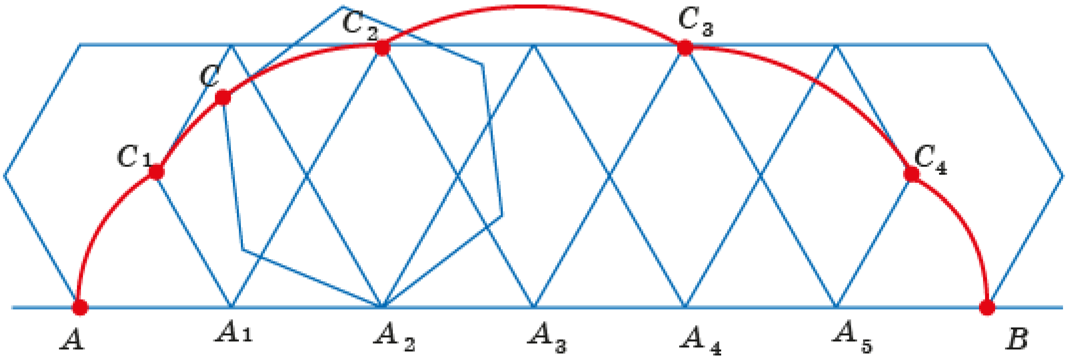


Fig. A5.8

18. A circle similar to the given one with a similarity ratio of 0.5. (Fig. A.5.9). **19.** A circle similar to the given one with a similarity ratio of 0.5. (Fig. A.5.10).

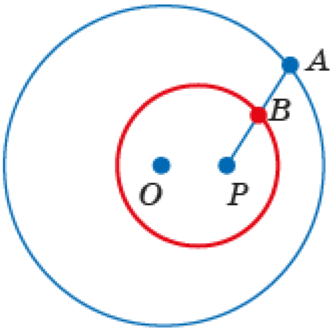


Fig. A5.9

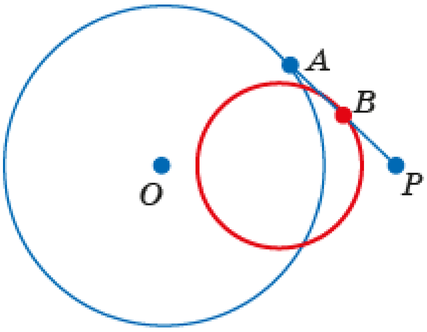


Fig. A5.10

20. Arc of a circle (рис. A5.11).

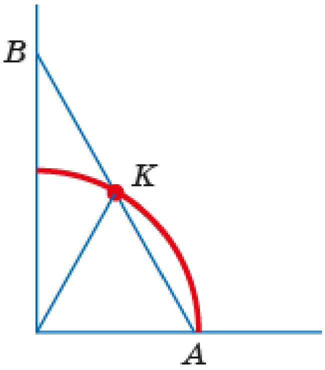


Fig. A5.11

6

2. a) Yes; b) No. **5.** Let the radius of the circle with center O be 1. The quadrilateral OAC_1Q_1 is an isosceles trapezoid (Fig. A6.1). Consequently, quadrilaterals OBC_1Q_1 and OBC_2Q_2 are parallelograms. Thus, $BC_1 = BC_2 = 2$. Therefore, this cardioid is a Pascal's snail.

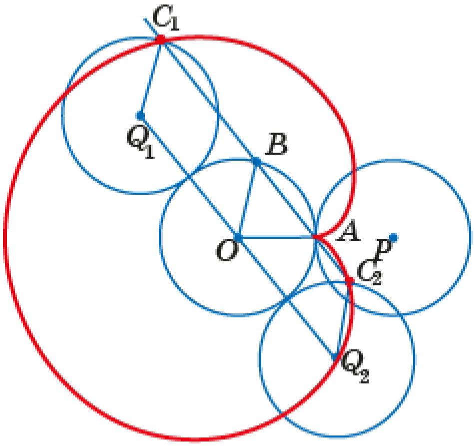


Fig. A6.1

6. A shortened cardioid is shown in Figure A6.2.

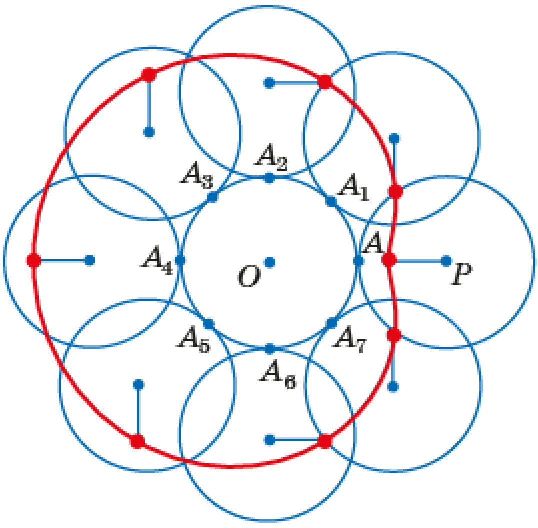


Fig. A6.2

7. To obtain a shortened cardioid in the GeoGebra software, do the following.

Create sliders for $t(0 \leq t \leq 2\pi)$ and $d(0 < d < 1)$.

Set, for example, $d = 0.6$.

Construct a circle with center $O(0,0)$and radius 1.

Construct a circle with center $P(2,0)$and radius 1.

Mark point $A(2 - d, 0)$.

Rotate points P and A around point O counterclockwise by an angle of $t$radians. Denote the resulting points as Q and B, respectively.

Rotate point B around point Q counterclockwise by an angle of $t$radians. Denote the resulting point as C.

Connect points Q and C with a segment.
In the properties of point C, select the "Show trace" option.

As you change the slider value, point C will leave a trace in the shape of a shortened cardioid (Fig. A6.3). If you select the "Animate" option in the slider's properties, you will see this trajectory in motion.

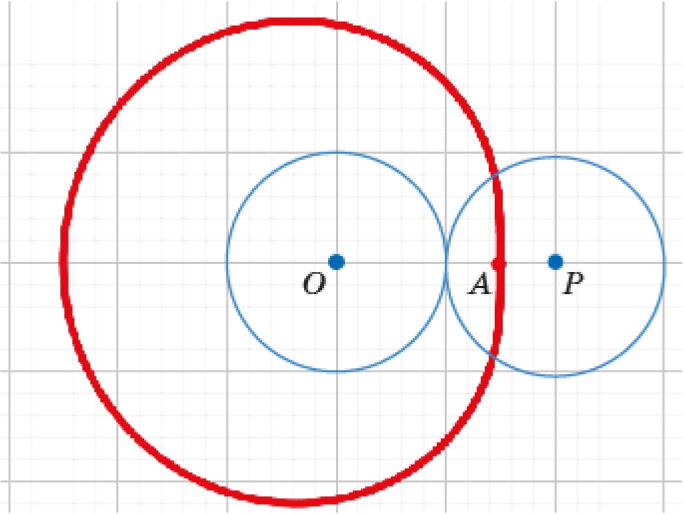


Fig. A6.3

8. Let the radius of the circle with center O be 1. The quadrilateral OAC_1Q_1 is an isosceles trapezoid (Fig. A6.4). Consequently, quadrilaterals OBC_1Q_1 and OBC_2Q_2 are parallelograms. Thus, $BC_1 = BC_2 = 2$. Therefore, this cardioid is a Pascal's snail.

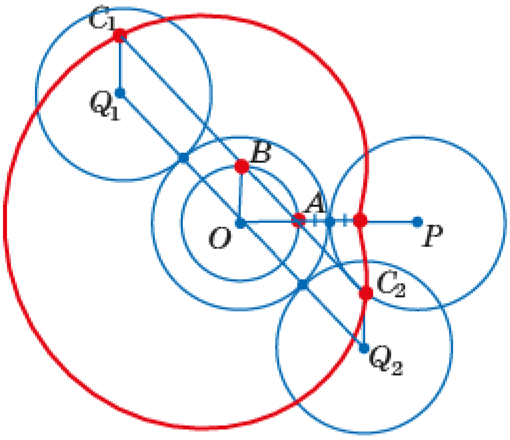


Fig. A6.4

9. A lengthened cardioid is shown in Figure A6.5.

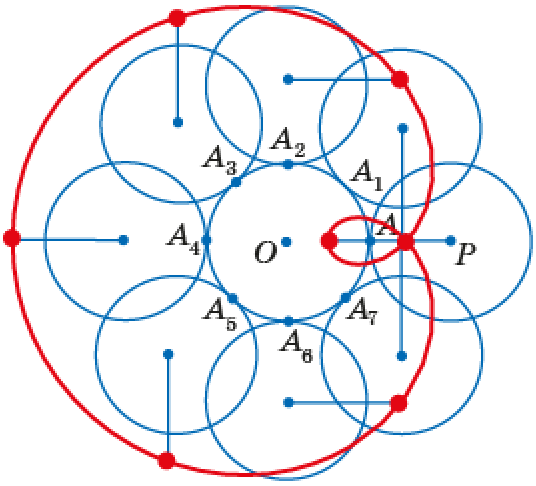


Fig. A6.5

10. To obtain a lengthened cardioid in the GeoGebra software, do the following.

Create sliders for $t(0 \le t \le 2\pi)$ and $d(1 < d < 2)$.

Construct a circle with center $O(0,0)$ and radius 1.

Construct a circle with center $P(2,0)$ and radius 1.

Mark point $A(2-d, 0)$.

Rotate points P and A around point O counterclockwise by an angle t. Denote the resulting points as Q and B, respectively.

Rotate point B around point Q counterclockwise by an angle t. Denote the resulting point as C.

Connect points Q and C with a segment. In the properties of point C, select the "Show trace" option.

As you change the slider value, point C will leave a trace in the shape of a lengthened cardioid (Fig. A6.6). If you select the "Animate" option in the slider's properties, you will see this trajectory in motion.

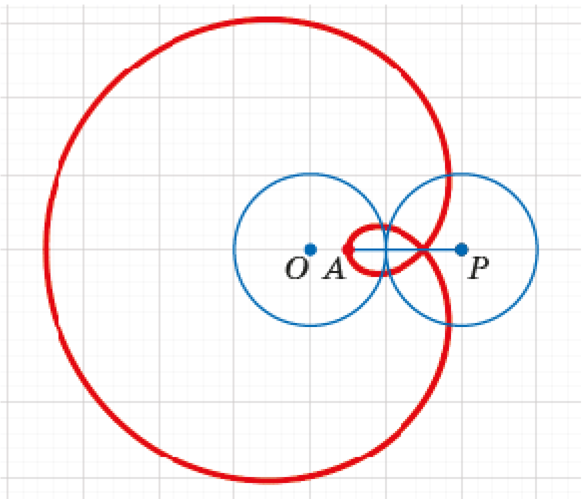


Fig. A6.6

11. Let the radius of the circle with center O be 1. The quadrilateral OAC_1Q_1 is an isosceles trapezoid (Fig. A6.7). Consequently, quadrilaterals OBC_1Q_1 and OBC_2Q_2 are parallelograms. Thus, $BC_1 = BC_2 = 2$. Therefore, this cardioid is a Pascal's snail.

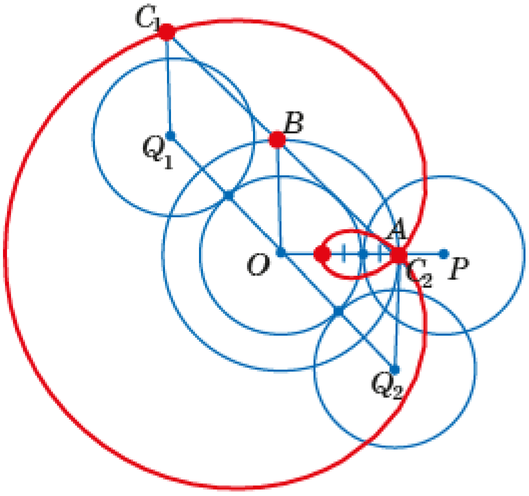


Fig. A6.7

12. To obtain an epicycloid with a radius ratio of 3:1, do the following.

Create a slider for $t(0 \le t \le 2\pi)$.

In the input bar, type $R = 3$ and $r = 1$.

Construct circles with radii $R = 3$ and $r = 1$ and centers $O(0,0)$, $P(R + r, 0)$.

Mark point $A(R, 0)$.

Rotate points P and A around point O counterclockwise by an angle of $t$radians.

Denote the resulting points as Q and B, respectively.

Construct a circle with center Q and radius r.

Rotate point B around point Q counterclockwise by an angle of $(R/r)t$ radians.

Denote the resulting point as C.

Connect points Q and C with a segment.

In the properties of point C, select the "Show trace" option. As you change the slider value, this point will leave a trace in the shape of an epicycloid (Fig. A6.8).

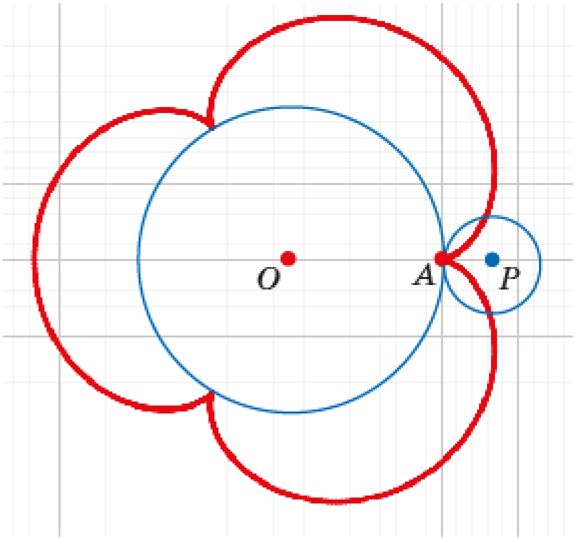


Fig. A6.8

If you select the "Animate" option in the slider's properties, you will see this trajectory in motion.

13. An epicycloid is shown in Figure A6.9.

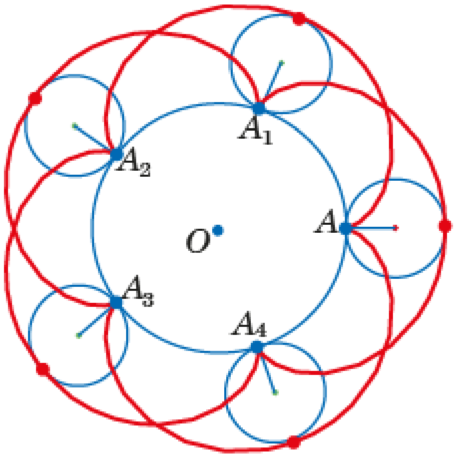


Fig. A6.9

To obtain an epicycloid with a radius ratio of 5:2, use the solution from exercise 6, setting $R = 5$, $r = 2$. The corresponding epicycloid is shown in Figure A6.10.

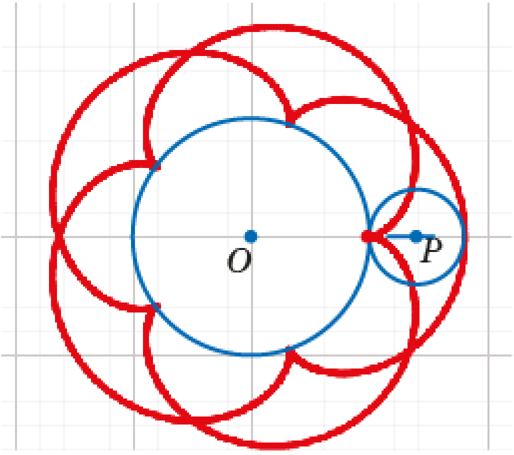


Fig. A6.10

14. To obtain an astroid in the GeoGebra software, do the following.

Create a slider for $t(0 \leq t \leq 2\pi)$.

Construct a circle with center $O(0,0)$and radius 4. Construct a circle with center $P(3,0)$and radius 1.

Mark point $C(3,0)$.

Rotate points P and Aaround point O counterclockwise by an angle of $t$radians. Denote the resulting points as Q and B, respectively.

Rotate point B around point Q clockwise by an angle of $4t$ radians. Denote the resulting point as C.

Connect points Q and C with a segment.

In the properties of point C, select the "Show trace" option. As you change the slider value, the point will leave a trace in the shape of an astroid (Fig. A6.11).

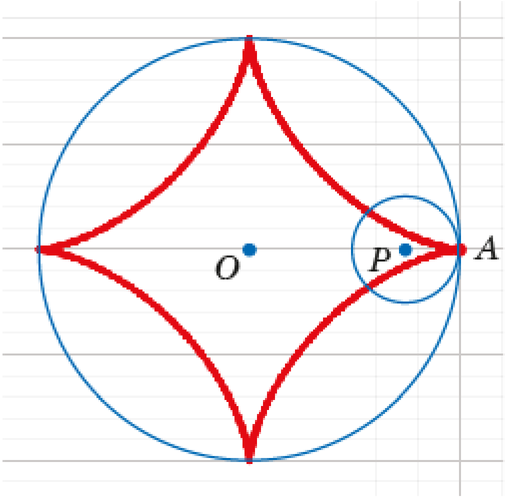


Fig. A6.11

If you select the "Animate" option in the slider's properties, you will see this trajectory in motion.

15. Steiner's curve is shown in Figure A6.12.

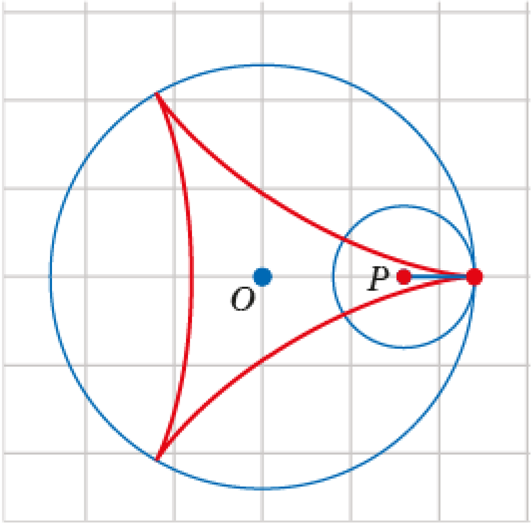


Fig. A6.12

To obtain Steiner's curve in the GeoGebra software, do the following.

Create a slider for $t (0 \leq t \leq 2\pi)$.

Construct a circle with center $O(0,0)$and radius 3.

Mark points $P(2, 0)$ and $A(3, 0)$.

Rotate points P and A around point O counterclockwise by an angle of $t$radians.

Denote the resulting points as Q and B, respectively.

Construct a circle with center Q and radius 1.

Rotate point B around point Q clockwise by an angle of $3t$ radians. Denote the resulting point as C.

Connect points Q and C with a segment.

In the properties of point C, select the "Show trace" option.

As you change the slider value, this point will leave a trace in the shape of Steiner's curve (Fig. A6.13).

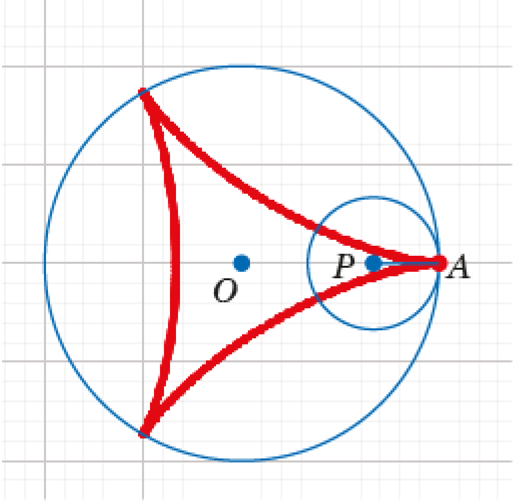


Fig. A6.13

If you select the "Animate" option in the slider's properties, you will see this trajectory in motion.

16. A hypocycloid is shown in Figure A6.14.

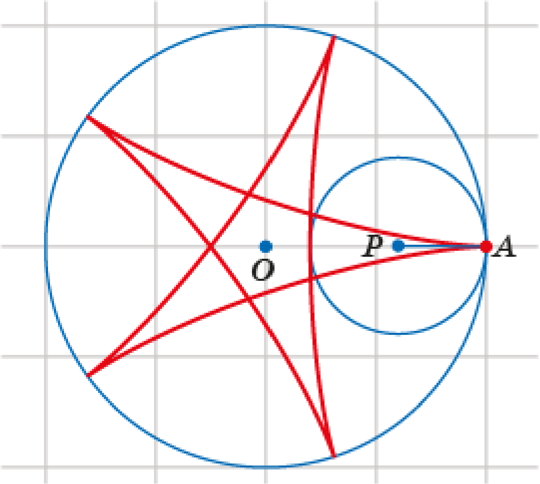


Fig. A6.14

To obtain a hypocycloid in the GeoGebra software, do the following.

Create a slider for $t (0 \leq t \leq 4\pi)$.

Construct a circle with center $O(0,0)$ and radius 5. Mark points $P(3,0)$ and $A(5,0)$.

Rotate points P and A around point O counterclockwise by an angle of t radians.

Denote the resulting points as Q and B, respectively.

Construct a circle with center Q and radius 2.

Rotate point B around point Q clockwise by an angle of $2.5t$ radians. Denote the resulting point as C.

Connect points Q and C with a segment. In the properties of point C, select the "Show trace" option. As you change the slider value, this point will leave a trace in the shape of Steiner's curve (Fig. A6.15).

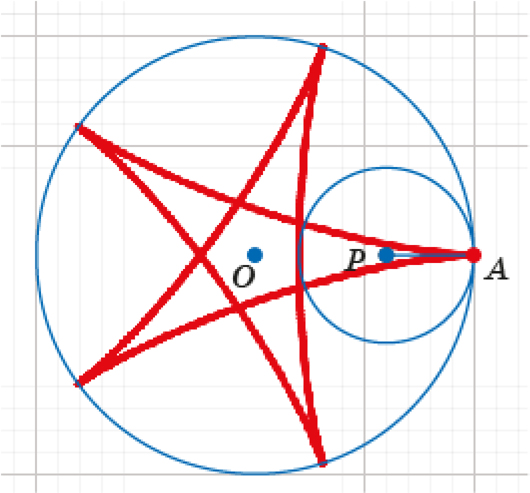


Fig. A6.15

If you select the "Animate" option in the slider's properties, you will see this trajectory in motion.

17. The desired curve is shown in Figure A6.16. Let's prove that this curve is a cardioid.

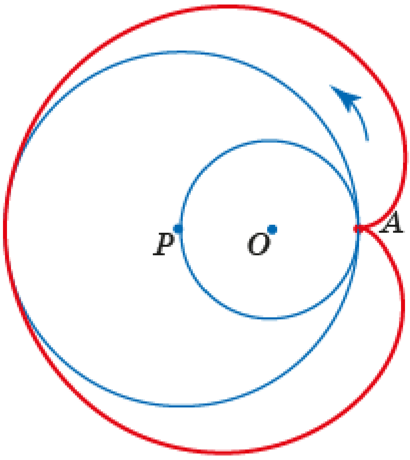


Fig. A6.16

Suppose the larger circle has rotated around the center O of the smaller circle by an angle $AOB = \alpha$ (Fig. A6.17). Let P' be the center of the resulting larger circle, and C be the point to which point $A(1, 0)$ has moved.

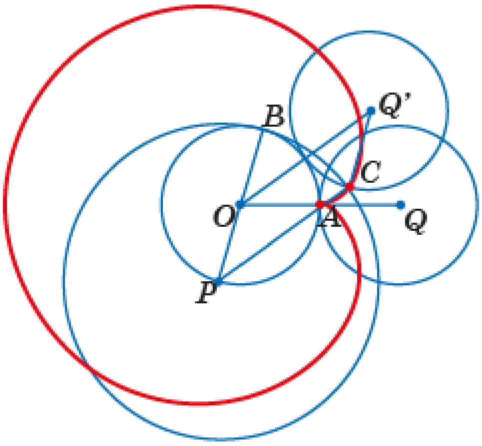


Fig. A6.17

Consider a unit circle with center Q', obtained by rotating a unit circle with center $Q(2, 0)$ by an angle $\alpha/2$. Then, in the quadrilateral $P'CQ'O$, sides $P'C$ and OQ' are equal and parallel. Therefore, this quadrilateral is a parallelogram. This means that sides $P'O$ and CQ' are equal, and angle $OQ'C$ is equal to $\alpha/2$. Consequently, point C belongs to the cardioid.

18. Let the large circle be rotated by an angle $AOB = \alpha$, P' be the center of the resulting large circle, and A' be the point to which point $A(1, 0)$ has moved (Fig. A6.18). Then $\overset{\frown}{A'B} = \overset{\frown}{AB} = \alpha$. Consequently, $\angle A'P'B = \frac{\alpha}{R}$. Through point O, draw a ray parallel to the line $P'A'$. Mark on it point C where it intersects the circle and point Q' such that $OQ' = R$. Then $\angle AOC = \alpha - \frac{\alpha}{R} = \frac{\alpha(R-1)}{R}$. Consequently, $\overset{\frown}{AC} = \frac{\alpha(R-1)}{R}$. Segments $P'A'$ and OQ' are equal and parallel. Therefore, quadrilateral

$P'A'Q'O$ is a parallelogram. This means that segments $A'Q'$ and $P'O$ are also equal and parallel. Consequently, point A' belongs to the circle with center Q' and radius R - 1. Since $\angle A'QO = \angle A'P'O = \frac{\alpha}{R}$, then $\overset{\frown}{A'C} = \frac{\alpha(R-1)}{R} = \overset{\frown}{AC}$. This means that point A' belongs to the trajectory of a point fixed on a circle of radius R - 1, rolling externally along the given circle.

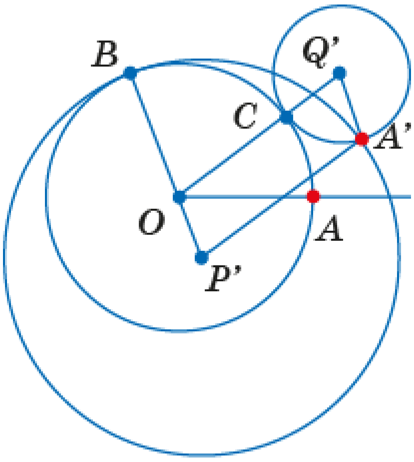


Fig. A6.18

20. The desired curve is shown in Figure A6.19.

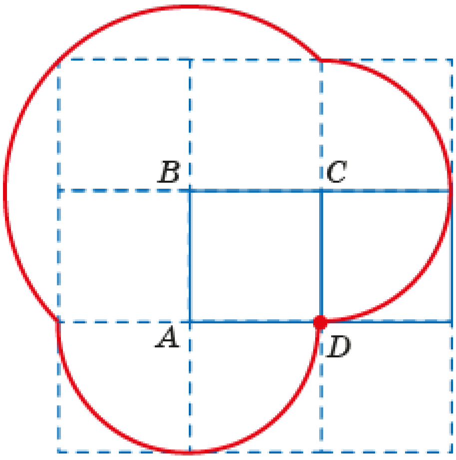


Fig. A6.19

22. The desired curve is shown in Figure A6.20.

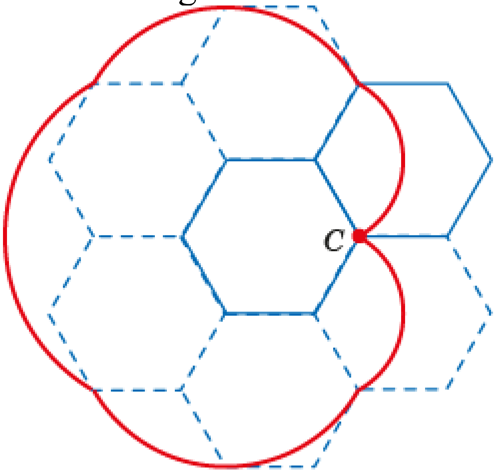


Fig. A6.20

24. The desired curve is shown in Figure A6.21. It consists of circular arcs.

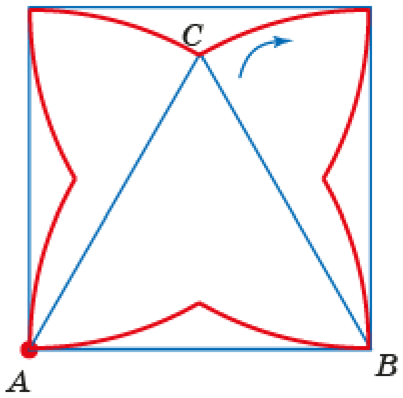


Fig. A6.21

25. The desired curve is shown in Figure A6.22. It consists of circular arcs.

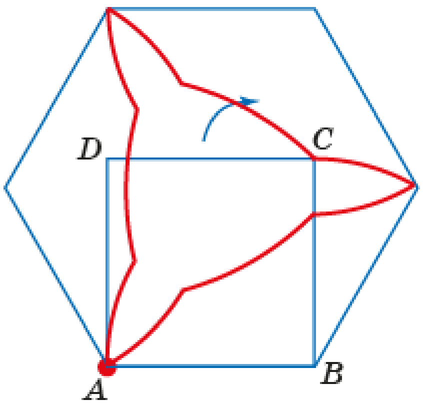


Fig. A6.22

7

1. The desired envelope is the parabola shown in red in the figure A7.1. It is similar to the parabola shown in blue in the same figure. The similarity ratio is 2.

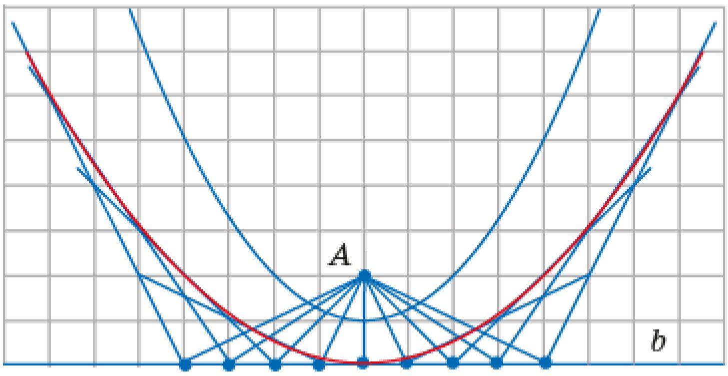


Fig. A7.1

2. The desired envelope is the ellipse shown in red in the figure A7.2. It is similar to the ellipse shown in blue in the same figure. The similarity ratio is 2.

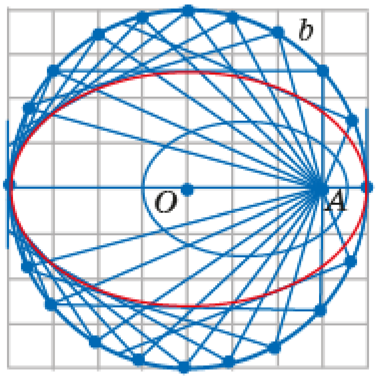


Fig. A7.2

3. The desired envelope is the hyperbola shown in red in the figure A7.3. It is similar to the hyperbola shown in blue in the same figure. The similarity ratio is 2.

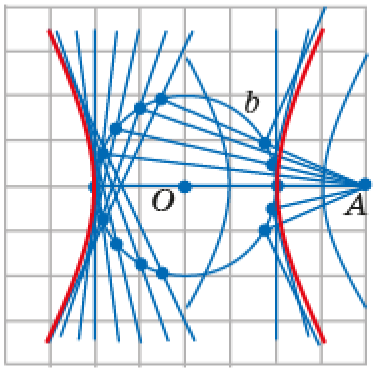


Fig. A7.3

4. The desired envelope consists of arcs of a cycloid (Fig. A7.4).

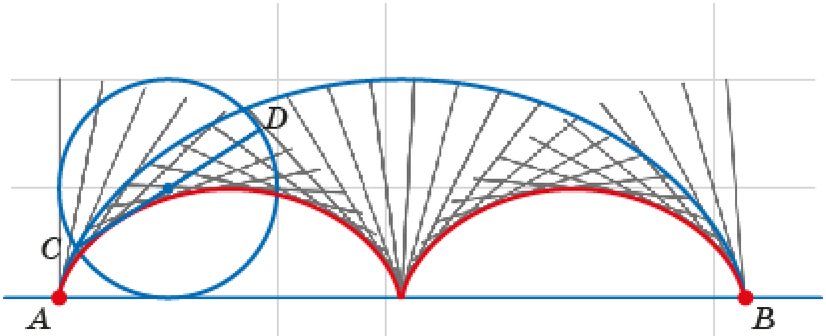


Fig. A7.4

5. The desired envelope is an arc of a cycloid (Fig. A7.5).

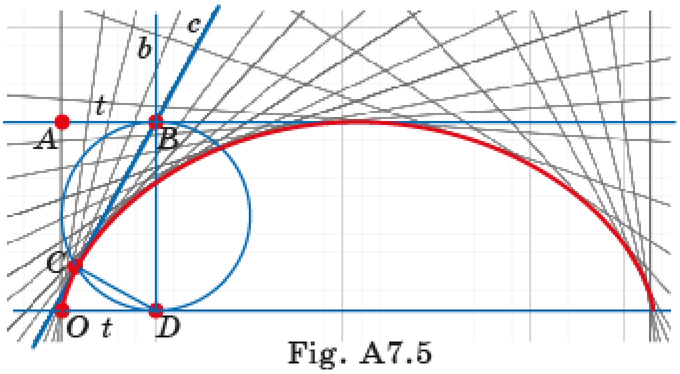
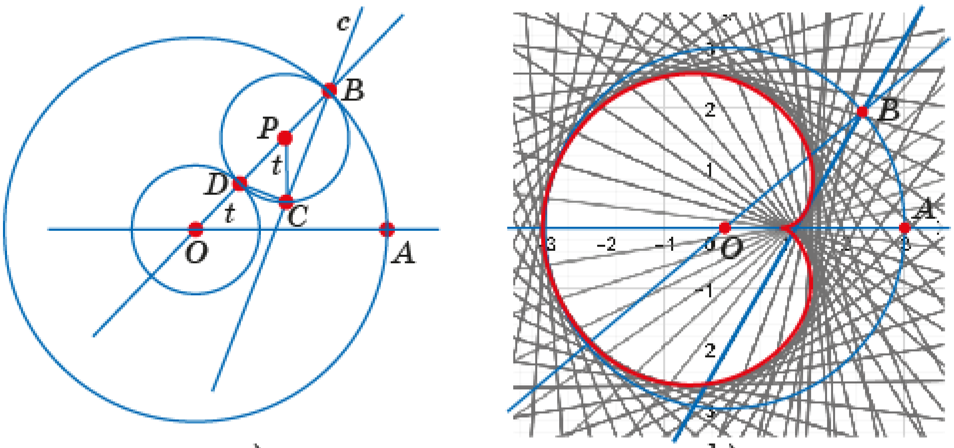


Fig. A7.5

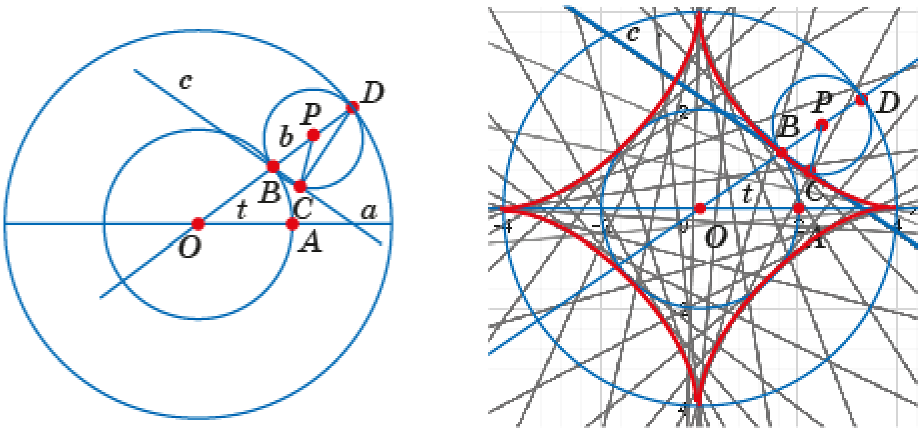
6. Consider circles with center *O* and radii 1 and 3. Let point *A* belong to the circle of radius 3. Point *B* is obtained by rotating point *A* around point *O* by an angle *t* counterclockwise. Line c is obtained by rotating line OB around point *B* by an angle *t*/2 counterclockwise. Consider the unit circle with center *P* that is internally tangent to the circle of radius 3 at point *B*. Let *D* be the point of tangency of this circle and the circle of radius 1, and *C* be the intersection point of this circle and line *c* (Fig. A7.6, a). Then point *C* belongs to the cardioid, and line *c* is tangent to this cardioid. Consequently, this cardioid is the envelope of the family of lines c (Fig. A7.6, b).



a) b)

Fig. A7.6

7. Consider circles with center *O* and radii 4 and 1. Let point *A* belong to the circle of radius 4. Point *B* is obtained by rotating point *A* around point *O* by an angle *t* counterclockwise. Line c is obtained by rotating line *OB* around point *B* by an angle 2*t* clockwise. Consider the unit circle with center *P* that is externally tangent to the circle of radius 4 at point *B*. Let *D* be the point of tangency of this circle and the circle of radius 1, and *C* be the intersection point of this circle and line *c* (Fig. A7.7, a). Then point *C* belongs to the astroid, and line *c* is tangent to this astroid. Consequently, this astroid is the envelope of the family of lines *c* (Fig. A7.7, b).

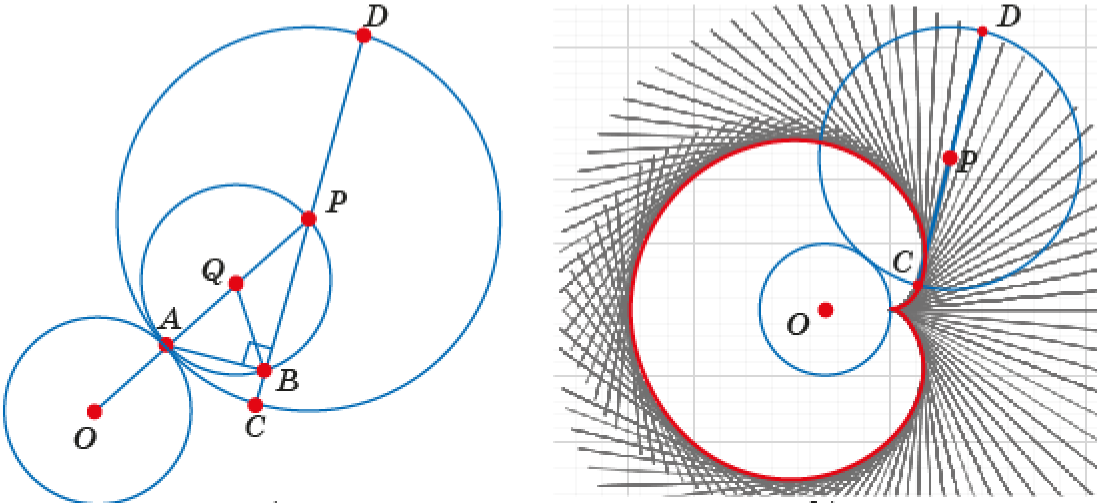


a) b)

Fig. A7.7

8. Let a circle with center *P* and radius 2 roll along the unit circle with center *O*. *A* is the point of tangency of these circles, *C* is a point fixed on the rolling circle.

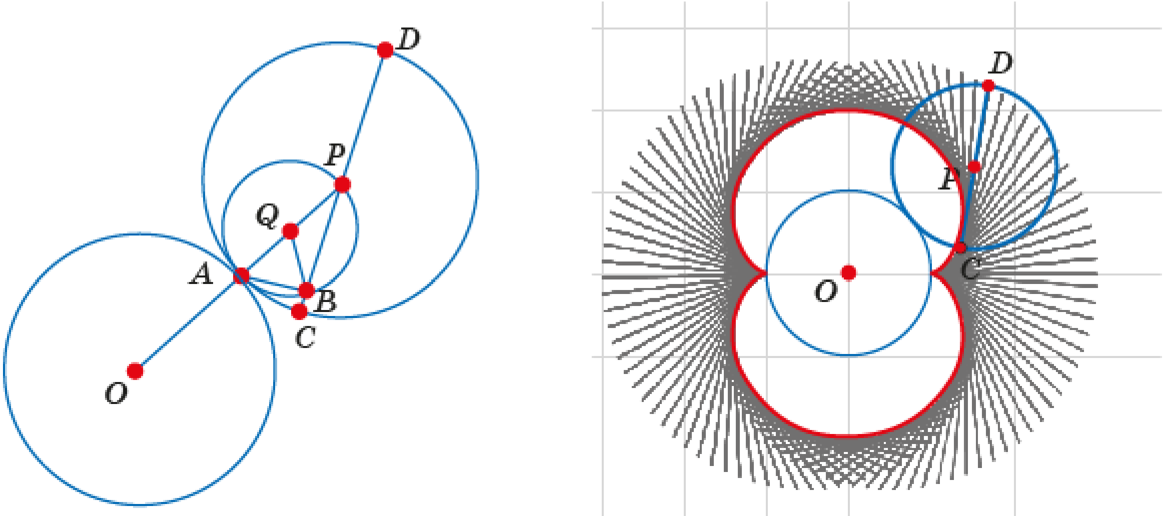
Consider the unit circle with center Q that is tangent to the unit circle with center O at point A (Fig. A7.8, a). Denote by B the intersection point of the radius PC and the rolling unit circle. Note that the arc length AB of this circle equals the arc length AC. Consequently, point B belongs to the cardioid. The diameter CD of the circle of radius 2 will be perpendicular to the segment connecting point B of the rolling unit circle to point A. Therefore, this diameter lies on the tangent to the cardioid. Hence, the envelope of the family of diameters CD is the cardioid (Fig. A7.8, b).



a) b)

Fig. A7.8

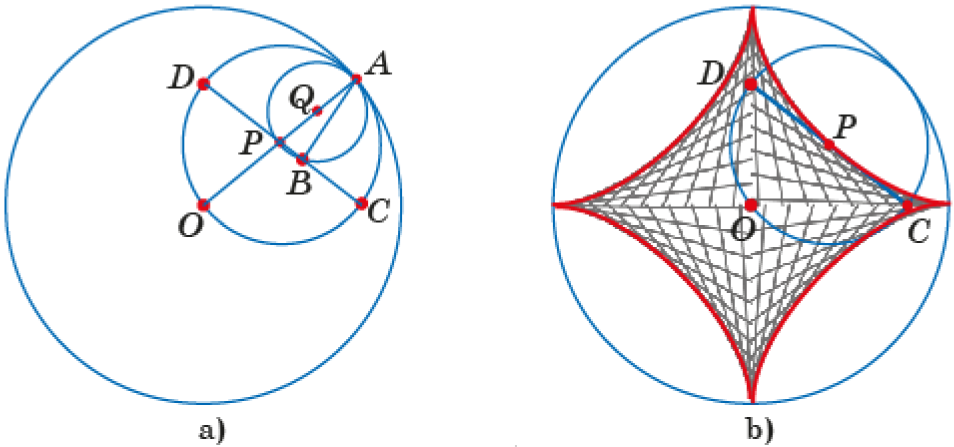
9. Let a unit circle with center P roll along the unit circle with center O. A is the point of tangency of these circles, C is a point fixed on the rolling circle, CD is a diameter. Consider the circle with center Q and radius 0.5 that is tangent to the unit circle with center O at point A (Fig. A7.9, a). Denote by B the intersection point of this circle and the segment PC. Note that the arc length AB equals the arc length AC. Consequently, point B belongs to the trajectory of a point fixed on a circle of radius 0.5 rolling externally along the unit circle, and line BP is tangent to this trajectory. Therefore, the envelope of the family of diameters CD is the trajectory of a point fixed on a circle of radius 0.5 rolling externally along a circle of radius 1 (Fig. A7.9, b).



a) b)

Fig. A7.9

10. Let a unit circle with center *P* roll inside a circle with center *O* and radius 2. *A* is the point of tangency of these circles, *C* is a point fixed on the rolling circle, *CD* is a diameter. Consider the circle with center *Q* and radius 0.5 that is tangent to the unit circle with center *O* at point *A* (Fig. A7.10, a). Denote by *B* the intersection point of this circle and the segment *PC*. Note that the arc length *AB* equals the arc length *AC*. Consequently, point *B* belongs to the trajectory of a point fixed on a circle with center *Q* rolling inside the circle with center *O*, and line *BP* is tangent to this trajectory. Therefore, the envelope of the family of diameters *CD* is the astroid (Fig. A7.10, b).



a) b)

Fig. A7.10

11. Similarly to how it was done for the previous problem, it is shown that the desired envelope is the Steiner curve (Fig. A7.11).

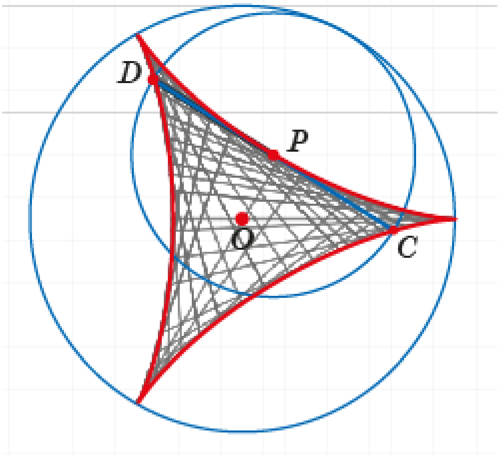


Fig. A7.11

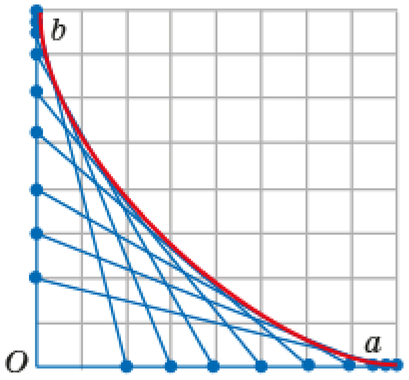


Fig. A7.12

12. Segments *AB* are diameters of circles of radius *c*/2 that are internally tangent to the circle of radius *c*. As was shown in the solution to Exercise 7, the envelope of such segments is the astroid (Fig. A7.12). **13.** The envelope is a segment of a parabola (Fig. A7.13).

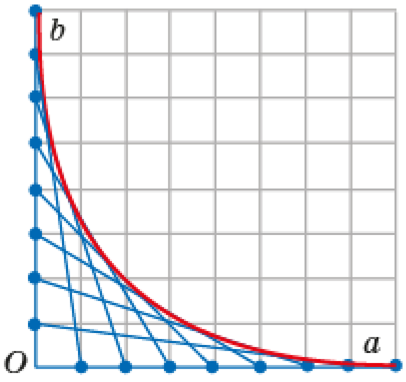


Fig. A7.13

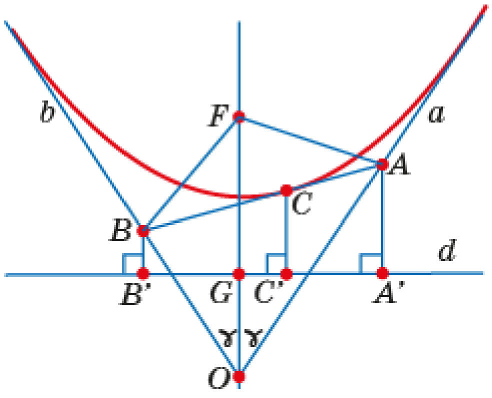


Fig. A7.14

It is true for any angle (Fig. A7.14). Indeed, from the properties of the parabola, it follows that the length of the segment *A'B'* is independent of the position of the point of tangency *C*. The equalities $OA = \frac{GA'}{\sin\gamma}$, $OB = \frac{GB'}{\sin\gamma}$, hold, from which the equality $OA + OB = \frac{A'B'}{\sin\gamma}$ follows. Therefore, it does not depend on the choice of the position of point *C*.

14. From the properties of the parabola it follows that the desired envelope is an arc of the parabola (Fig. A7.15). It is true for any angle *AOB* and *ACB* such that $\angle aOb + \angle ACB = 180°$.

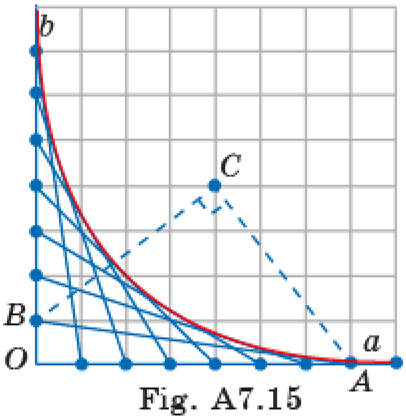


Fig. A7.15

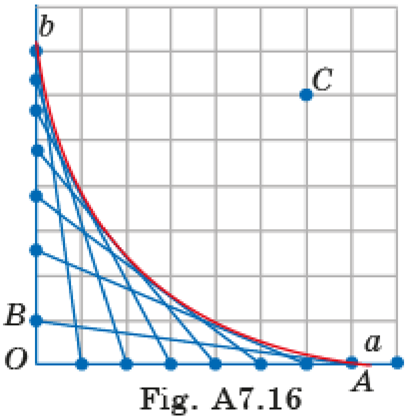


Fig. A7.16

15. From the properties of the ellipse it follows that the desired envelope is an arc of the ellipse (Fig A7.16). It is true for any angle *AOB* and *ACB* such that $\angle aOb + \angle ACB < 180°$.

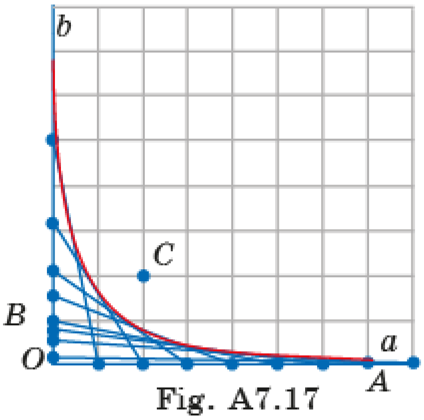


Fig. A7.17

16. From the properties of the hyperbola, it follows that the desired envelope is an arc of the hyperbola (Fig. A7.17). It is true for any angle AOB and ACB such that $\angle aOb + \angle ACB > 180°$.

17. Let a unit circle pass through point A of a given unit circle with center O, and let its center P belong to the given circle. Consider the unit circle with center Q that is tangent to the given circle at point P (Fig. A7.18). Denote by B the intersection point of ray OP with this circle, and by C the intersection point of the circles with centers P and Q. Note that the arc length PA equals the arc length PC. Consequently, point C belongs to the trajectory of a point fixed on a circle with center Q rolling along the circle with center O. This trajectory is a cardioid. Line BC is tangent to the circle with center P and to this cardioid. Therefore, this cardioid is tangent to the circles with centers P, meaning it is the envelope of this family of circles.

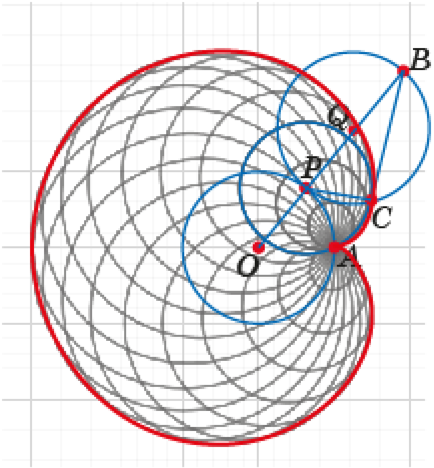


Fig. A7.18

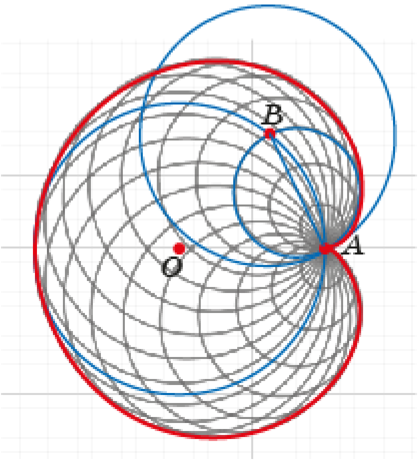


Fig. A7.19

18. Circles with diameter AB are obtained from circles with centers B passing through point A by a homothety with center A and factor 0.5. Therefore, the envelope of the family of circles with diameters AB is obtained from the envelope of the family of circles with centers B. As shown in the solution to the previous exercise, the envelope of the family of circles with centers B is a cardioid. Consequently, the desired envelope of the family of circles with diameters AB is the cardioid (Fig. A7.19).

19. As follows from the solution to Exercise 10, the desired envelopes are two arcs of an astroid (Fig. A7.20).

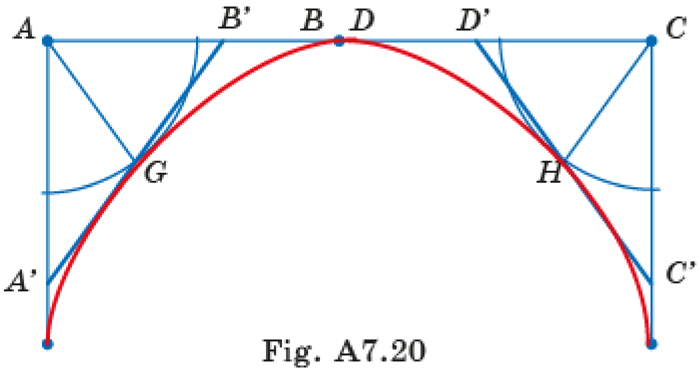


Fig. A7.20

8

1. 120°.

2. Through point E_1 on arc D_1D_2 and point A, draw a straight line. Its intersection point with arc D_4D_5 is denoted as E_4. The tangents to the curve at points E_1 and E_4 will be perpendicular to segment E_1E_4. Consequently, the length of this segment will be the width h of the curve in the direction of line E_1E_4, $h = r + a + r - b - c = 2r + a - b - c$. Clearly, this value does not depend on the choice of point E_1 on arc D_1D_2 (Fig. A8.1).

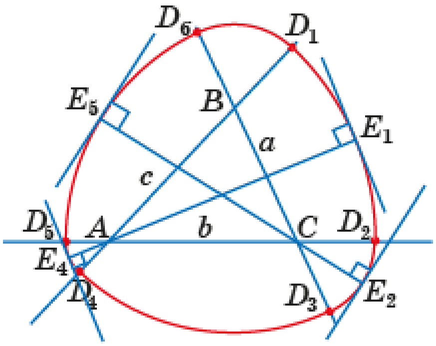


Fig. A8.1

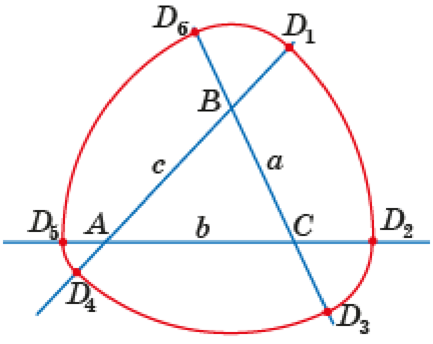


Fig. A8.2

Now consider point E_2 on arc D_2D_3. Through it and point C, draw a straight line. Its intersection point with arc D_5D_6 is denoted as E_5. The tangents to the curve at points E_2 and E_5 will be perpendicular to segment E_2E_5. Consequently, the length of this segment will be the width of the curve in the direction of line E_2E_5. It equals $r - b + a + r - c = 2r + a - b - c$. Clearly, this value does not depend on the choice of point E_2 on arc D_2D_3. In a similar manner, it is shown that for points E_3 on arc D_3D_4, the width of the curve is also equal to $2r + a - b - c$.

3. Recall that the length l of an arc of a circle with radius R and central angle φ is given by the formula $l = \varphi R$. Let the angles of triangle ABC be α, β, γ respectively. Then the length of arc D_1D_2 equals αr, and the length of arc D_4D_5 equals $\alpha(a + r - b - c)$. Their sum equals αh. Similarly, the sum of the lengths of arcs D_2D_3 and D_5D_6 equals γh, and the sum of the lengths of arcs D_3D_4 and D_6D_1 equals βh. Thus, the length of the entire curve equals $\alpha h + \beta h + \gamma h = (\alpha + \beta + \gamma)h = \pi h$, i. e., it equals the circumference of a circle with diameter h (Fig. A8.2).

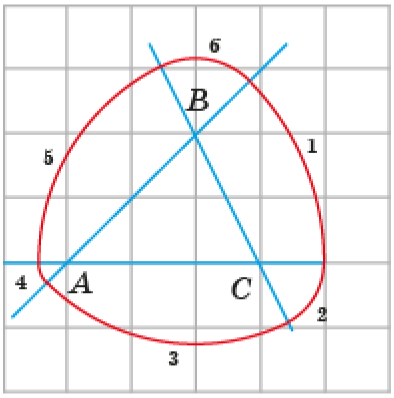


Fig. A8.3

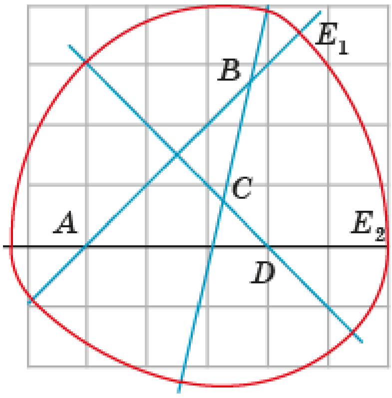


Fig. A8.4

4. $\frac{1+\sqrt{5}}{2}$. **5.** 144°. **6.** The required curve is shown in Figure A8.3. It consists of arcs 1, 2, 3, 4, 5, 6 of circles with centers at A, C, B, A, C, B respectively. **7.** The required curve is shown in Figure A8.4, $h = a + 2r + b + c - d$. **8.** The required curve is shown in Figure A8.5, $h = a + 2r + b + d - c - e$.

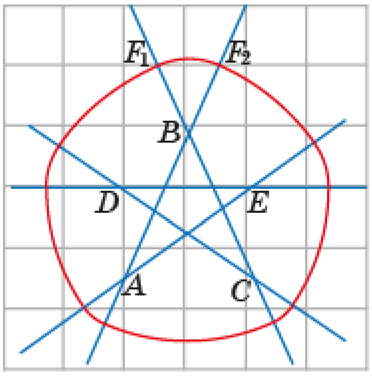


Fig. A8.5

9. The corresponding file is available on the website vasmirnov.ru in the "Geometry with GeoGebra" section. **10.** The corresponding file is available on the website vasmirnov.ru in the "Geometry with GeoGebra" section.

9

1. а) $y = 0$; б) $x = 0$. **2.** а) $x = a$; б) $y = b$. **3.** $x - y - 1 = 0$. **4.** а) $x - y - 1 = 0$; б) $3x + 2y - 8 = 0$. **5.** $2x - y + 5 = 0$. **6.** $x - 2y + 10 = 0$. **7.** а) $y = 3$; б) $x = -1$; в) $2x - y + 5 = 0$. **8.** $2x + y - 3 = 0$. **9.** а) $x + y - 1 = 0$; б) $2x - 3y + 8 = 0$. **10.** а) $y = x$; б) $y = 2x$; в) $y = 0{,}5x$; г) $y = -x$; д) $y = -2x$; е) $y = -0{,}5x$. **11.** а) $1\frac{1}{2}$; б) -2. **12.** $x + y - 1 = 0$. **13.** $x - 2y + 7 = 0$, $\vec{n}(1, -2)$. **14.** a_1: $x + 3y - 3 = 0$; a_2: $2x - y + 2 = 0$. **15.** а) 1, 3; б) 2, 4. **16.** а) (-1, -2); б) (7, 3). **17.** а) $x + y - 5 = 0$; б) $2x - y + 1 = 0$. **18.** а) $4x - 7y + 14 = 0$, $4x + 5y - 13 = 0$, $4x - y - 6 = 0$, $\left(2\frac{1}{3}, 3\frac{1}{3}\right)$; б) $x - 4y + 8 = 0$, $3x - 4y - 20 = 0$, $x - 3 = 0$, $\left(3, 2\frac{3}{4}\right)$; в) $2x - 8y + 25 = 0$, $6x + 8y - 41 = 0$, $x - 2 = 0$; г) $(2, 3\frac{5}{8})$. **19.** а), б) 90°. **20.** а) $\frac{\sqrt{2}}{10}$; б) $\frac{\sqrt{2}}{2}$. **21.** а) 2,4; б) 1,4. **22.** а) $ax - by + c = 0$; б) $-ax + by + c = 0$; в) $ax + by - c = 0$.

10

1. This equation is reduced to the form $\left(x + \frac{a}{2}\right)^2 + \left(y + \frac{b}{2}\right)^2 = \frac{a^2}{4} + \frac{b^2}{4} - c$. If $c < \frac{a^2}{4} + \frac{b^2}{4}$, then this equation defines a circle with center at point $O\left(-\frac{a}{2}, -\frac{b}{2}\right)$and radius $R = \frac{\sqrt{a^2+b^2-4c}}{2}$. **2.** $\left(0, \frac{1}{4}\right)$, $y = -\frac{1}{4}$. **3.** A parabola with focus $\left(\frac{1}{4}, 0\right)$ and directrix $x = -\frac{1}{4}$ (Fig. A10.1). **4.** The equation $y = ax^2 + bx + c$is reduced to the form $y = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$. It defines a parabola with focus $\left(-\frac{b}{2a}, \frac{1}{4a}\right)$ and directrix $y = \frac{4ac-b^2-a}{4a^2}$. **5.** An ellipse with foci $F_1\left(0, -\frac{\sqrt{2}}{2}\right)$, $F_2\left(0, \frac{\sqrt{2}}{2}\right)$ (Fig. A10.2).

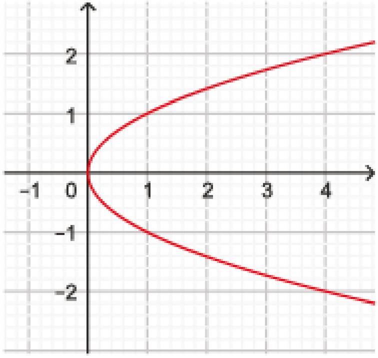


Fig. A10.1

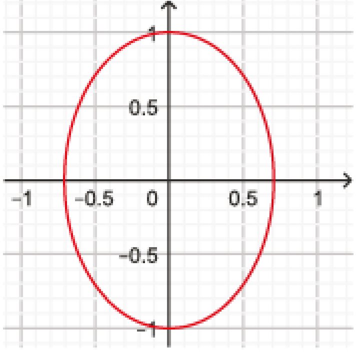
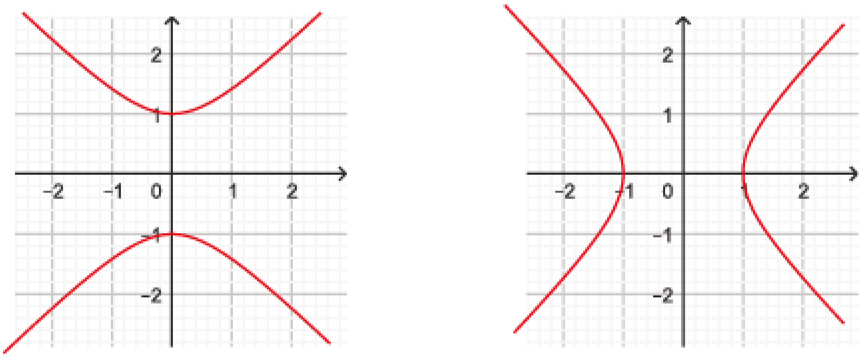


Fig. A10.2

6. The equation $ax^2 + bx + cy^2 + dy + e = 0$is reduced to the form $a\left(x + \frac{b}{2a}\right)^2 + c\left(y + \frac{d}{2c}\right)^2 = \frac{b^2}{4a} + \frac{d^2}{4c} - e$, which defines an ellipse. **7.** Let $A_1(-a, 0)$, $A_2(a, 0)$, $C(x, y)$. $A_1C^2 = (x+a)^2 + y^2$, $A_2C^2 = (x-a)^2 + y^2$. We have the equation $(x+a)^2 + y^2 + (x-a)^2 + y^2 = c^2$, which simplifies to $x^2 + y^2 = \frac{c^2}{2} - a^2$. This is the equation of a circle with center at $O(0, 0)$ and radius $R = \sqrt{\frac{c^2}{2} - a^2}$.**8.** The curve defined by the equation $y^2 - x^2 = 1$is shown in Figure A10.3, a. Rotate it around the origin by 90°. We obtain the curve shown in Figure A10.3, b. Its equation has the form $x^2 - y^2 = 1$. Consequently, this curve is a hyperbola.

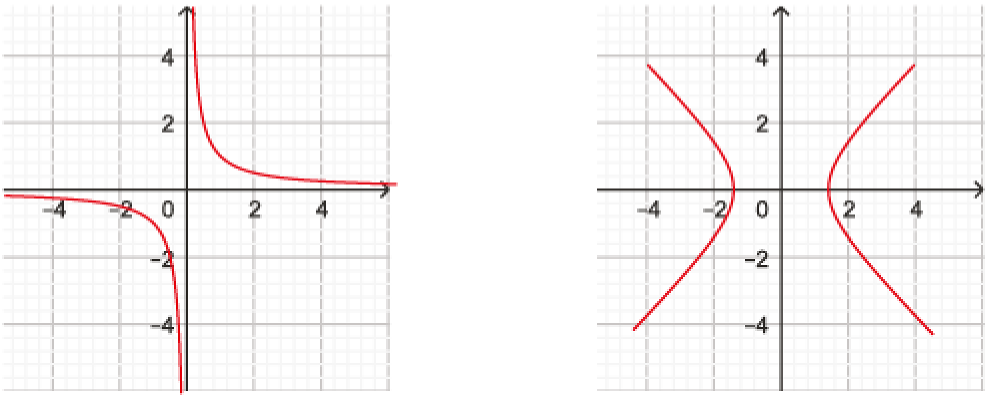


a)

Fig. A10.3

б)

9. The equation $ax^2 + bx - cy^2 - dy + e = 0$is reduced to the form $a\left(x + \frac{b}{2a}\right)^2 - c\left(y + \frac{d}{2c}\right)^2 = \frac{b^2}{4a} - \frac{d^2}{4c} - e$, which defines a hyperbola. **10.** The curve defined by the equation $xy = 1$ is shown in Figure A10.4, a. Rotate it around the origin by 45°clockwise. We obtain the curve shown in Figure A10.4, b. Its equation has the form $\frac{x^2}{2} - \frac{y^2}{2} = 1$. Consequently, this curve is a hyperbola.



a) б)

Fig. A10.4

11. Hyperbola. **12.** Ellipse. **13.** For the coordinates of points of this curve, the equality $x^2 + y^2 = k^2((x-b)^2 + y^2)$ holds. It is reduced to the form $\left(x - \frac{k^2 b}{1-k^2}\right)^2 + y^2 = \frac{k^2 b^2}{(1-k^2)^2}$ and defines a circle, which is called the Circle of Apollonius. Figure A10.5 shows such a circle for $b = 3$, $k = 2$.

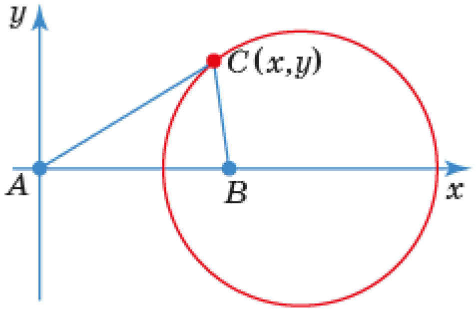


Fig. A10.5

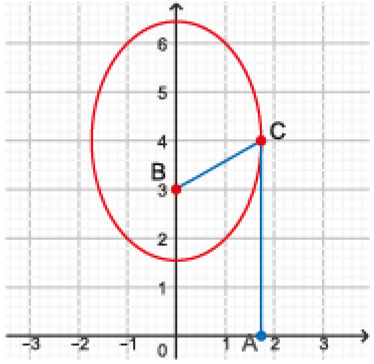


Fig. A10.6

14. For the coordinates of points of this curve, the equality $y^2 = k^2(x^2 + (y-b)^2)$ holds. It is reduced to the form $x^2 + \left(1 - \frac{1}{k^2}\right)\left(y - \frac{k^2 b}{k^2-1}\right)^2 = \frac{b^2}{k^2-1}$. If $k > 1$, it defines an ellipse. Figure A10.6 shows an ellipse for which $k = 2$, $b = 3$.

15. If $0 < k < 1$, then the equation $x^2 + \left(1 - \frac{1}{k^2}\right)\left(y - \frac{k^2 b}{k^2-1}\right)^2 = \frac{b^2}{k^2-1}$ defines a hyperbola. Figure A10.7 shows a hyperbola for which $k = 0.5$, $b = 3$.

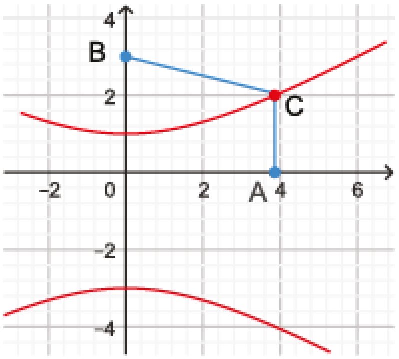


Fig. A10.7

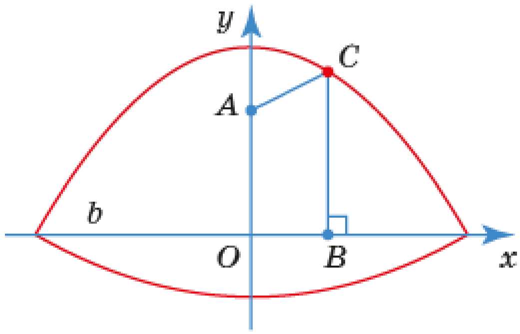


Fig. A10.8

16. Consider point $A(0,a)$and line $b$given by the equation $y = 0$. Find the locus of points $C(x,y)$for which the sum of the distances to point $A$and line $b$is equal to a given number $c > a$ (Fig. A10.8). We have the equation $\sqrt{x^2 + (y - a)^2} + | y | = c$, from which we obtain two equations: $2(c - a)y = c^2 - a^2 - x^2$ $(y \geq 0)$; $2(c + a)y = -c^2 + a^2 + x^2 (y \leq 0)$. These equations define parts of two parabolas. To obtain this locus of points in the GeoGebra software, create sliders a, c, and t. Construct point $A(0, a)$and line $b$given by the equation $y = 0$. Construct a line given by the equation $y = t$. Construct a circle with center A and radius $c -| t |$. Mark the intersection points of the constructed circle and line. In the properties of these points, select the "Show Trace" option. When changing the value of slider t, these points will leave a trace in the form of parts of two parabolas (Fig. A10.9).

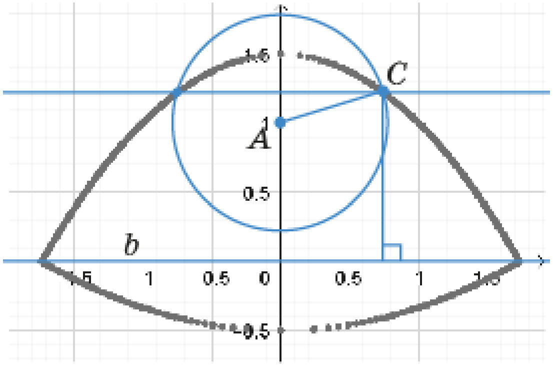


Fig. A10.9

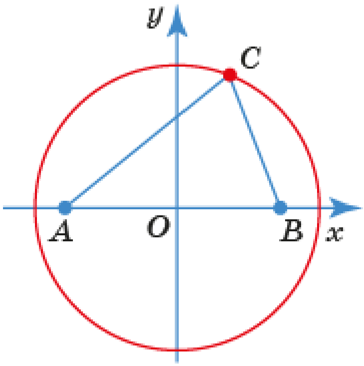


Fig. A10.10

17. Consider points $A(-a, 0)$and $B(a, 0)$. Find the locus of points $C(x, y)$for which the sum of the squares of the distances to points A and B is equal to a given number $c > 0$ (Fig. A10.10). We have the equation $(x + a)^2 + y^2 + (x - a)^2 + y^2 = c$, from which we obtain the equation $x^2 + y^2 = \frac{c}{2} - a^2$. If $\frac{c}{2} > a^2$, then this equation defines a circle. To obtain this locus of points in the GeoGebra software, create sliders a, c, and t. Construct points $A(-a, 0)$and $B(a, 0)$. Construct a circle with center A and radius t. Construct a circle with center B and radius $\sqrt{c - t^2}$. Mark the intersection points of the constructed circles. In the properties of these points, select the "Show Trace" option. When changing the value of slider t, these points will leave a trace in the form of a circle (Fig. A10.11).

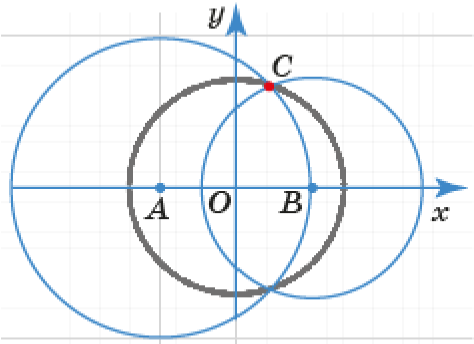


Fig. A10.11

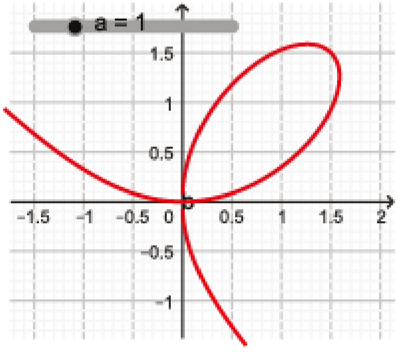


Fig. A10.12

18. Similarly to the solution of the previous exercise, it is proved that the desired locus of points is the line $4ax = c$. **19.** Create a slider a. In the "Input" bar, type $x^3 + y^3 - 3axy = 0$. We obtain the Folium of Descartes (Fig. A10.12). **20.** Introduce a coordinate system for which $F_1(-a, 0)$, $F_2(a, 0)$ (Fig. A10.13).

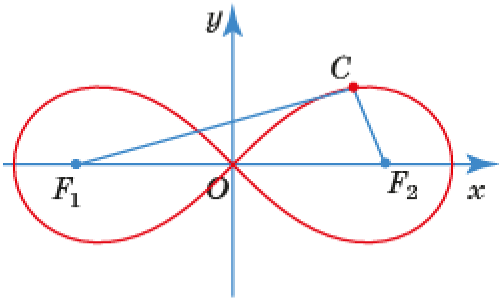


Fig. A10.13

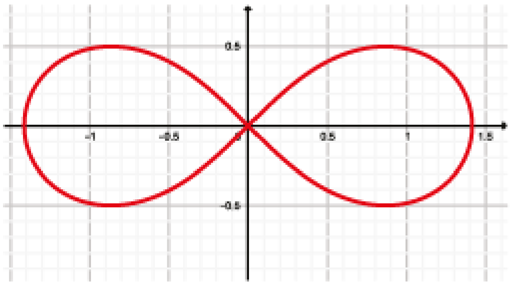


Fig. A10.14

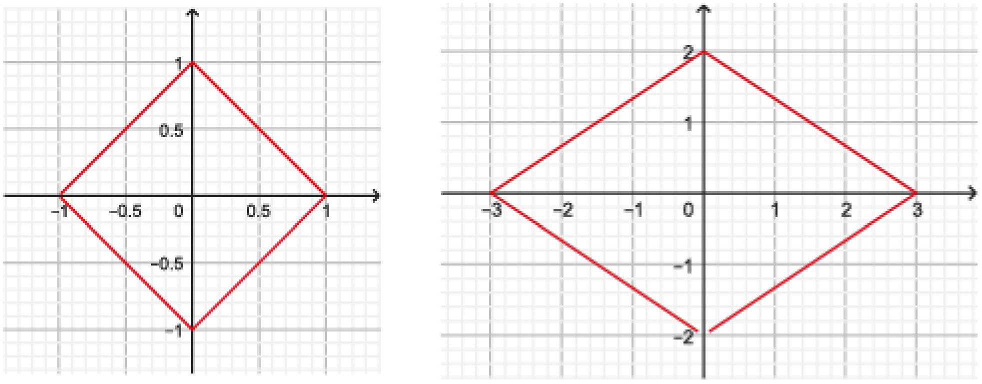
Let point $C(x,y)$belong to the lemniscate. The distances from point C to points F_1, F_2 are respectively $\sqrt{(x+a)^2+y^2}$, $\sqrt{(x-a)^2+y^2}$. Since the product of these distances equals a^2, we obtain the equation

$$\sqrt{(x+a)^2+y^2} \cdot \sqrt{(x-a)^2+y^2} = a^2,$$

which, upon transformation, yields the desired equation of the lemniscate

$$(x^2+y^2)^2 = 2a^2(x^2-y^2).$$

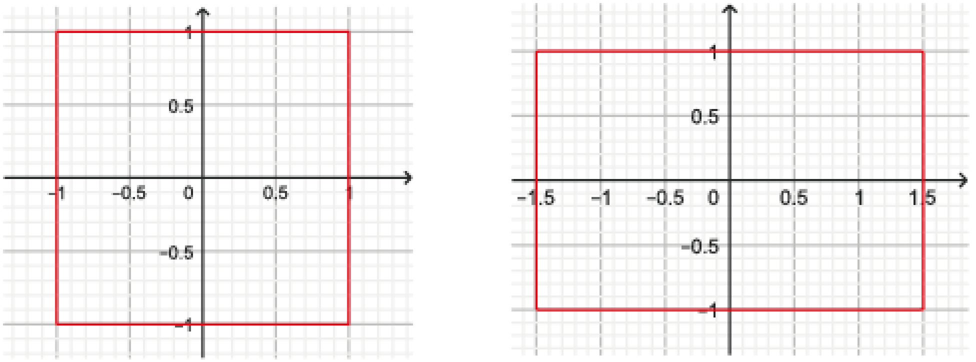
To obtain the lemniscate defined by this equation in the GeoGebra software, create a slider a. In the "Input" bar, type (x^2+y^2)^2=2a^2(x^2–y^2) and press "Enter". The lemniscate will appear on the screen. The value of a can be changed. The shape of the lemniscate will also change accordingly. Figure A10.14 shows the Bernoulli lemniscate for which $a = 1$. **21.** Figure A10.15.



a) b)

Fig. A10.15

22. Figure A10.16.



a) б)

Fig. A10.16

23. Figure A10.17.

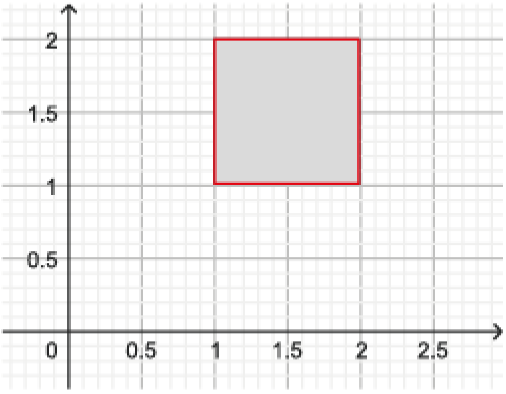


Fig. A10.17

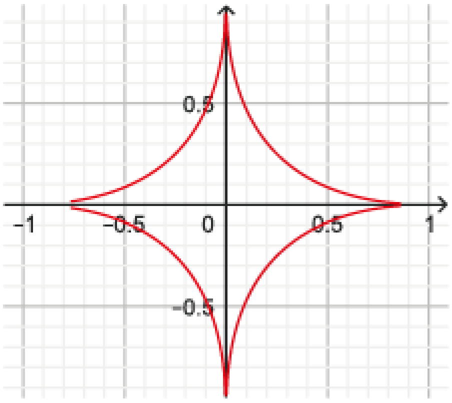


Fig. A10.18

24. Figure A10.18. **25.** Figure A10.19.

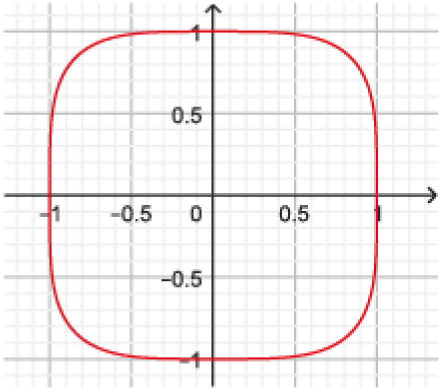


Fig. A10.19

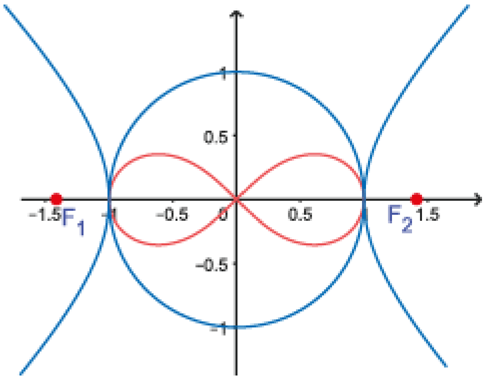


Fig. A10.20

11

1. $\begin{cases} x = x_0 + Rt, \\ y = y_0 + Rt. \end{cases}$ **2.** $\begin{cases} x = x_1 + (x_2 - x_1)t, \\ y = y + (y_2 - y_1)t. \end{cases}$ **3.** $\begin{cases} x = 5t, \\ y = 5\sqrt{3}t. \end{cases}$ **4.** $\begin{cases} x = x_1 + \frac{(x_2 - x_1)}{t_2 - t_1}(t - t_1), \\ y = y_1 + \frac{(y_2 - y_1)}{t_2 - t_1}(t - t_1). \end{cases}$ **5**. For the ellipse defined by the parametric equations $\begin{cases} x = a\cos t\,, \\ y = b\sin t\,, \end{cases}$ let's create sliders for a and b. In the "Input" bar, type: Curve(acos(t), bsin(t), t, 0, 2Pi) and press "Enter". This will produce an ellipse. Changing the values of the sliders will change the shape of the ellipse. Figure A11.1 shows the case *a* = 2, *b* = 1.

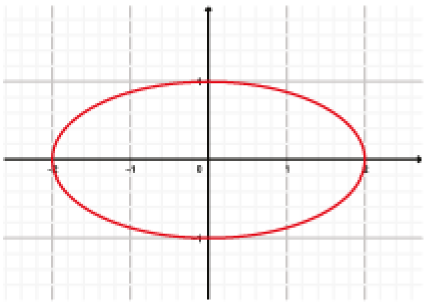


Fig. A11.1

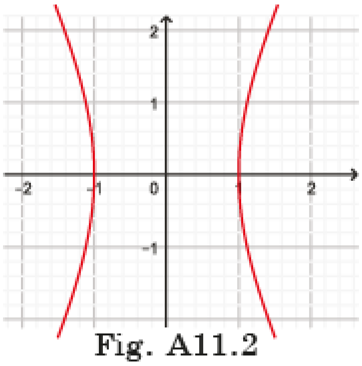


Fig. A11.2

6. For the hyperbola defined by the parametric equations $\begin{cases} x = a\,\mathrm{ch}\,t\,, \\ y = b\,\mathrm{sh}\,t\,, \end{cases}$ let's create sliders for a and b. In the "Input" bar, type: Curve(acosh(t), bsinh(t), t, -2, 2). This will produce one branch of the hyperbola. To obtain the second branch of the hyperbola, simply type Curve(-acosh(t), bsinh(t), t, -2, 2). Changing the values of the sliders will change the shape of the hyperbola. Figure A11.2 shows the case a = 1, b = 2.

7. Create a slider for a. In the "Input" bar, type: Curve(3a*t/(1+t^3), 3a*t^2/(1+t^3), t, -40, 40) and press "Enter". This will produce the Folium of Descartes. Changing the value of the slider will change the shape of the Folium of Descartes. Figure A11.3 shows the case for $a = 1$.

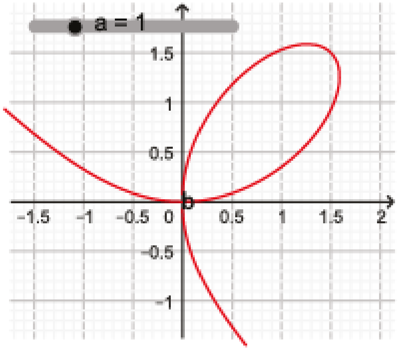


Fig. A11.3

8. Let the unit circle initially be tangent to the origin. A point is fixed on the extension of the circle's radius and is at a distance $d > 1$ from its center. Suppose the circle rolls along the Ox axis and moves to point Q. The center of the circle moves to point P. The point fixed on the circle moves to point C (Fig. A11.4).

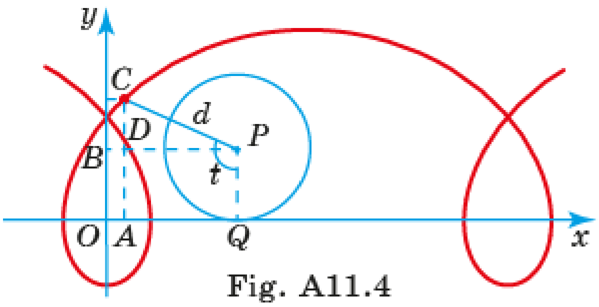


Fig. A11.4

Let $\angle CPQ = t$. For the coordinates x, y of point C we have

$$\begin{cases} x = OQ - AQ = t - d \cdot \cos{(t - 90°)} = t - d \cdot \sin t, \\ y = PQ + CD = 1 + d \cdot \sin{(t - 90°)} = 1 - d \cdot \cos t. \end{cases}$$

Thus, the parametric equations of the prolate cycloid are the equations

$$\begin{cases} x = t - d \cdot \sin t, \\ y = 1 - d \cdot \cos t. \end{cases}$$

To obtain a prolate cycloid in the GeoGebra software, you need to create a slider *d* > 1. In the "Input" bar, type: Curve(t-d*sin(t), 1-d*cos(t), t, 0, 2Pi) and press "Enter". The result will be a prolate cycloid (Fig. A11.5).

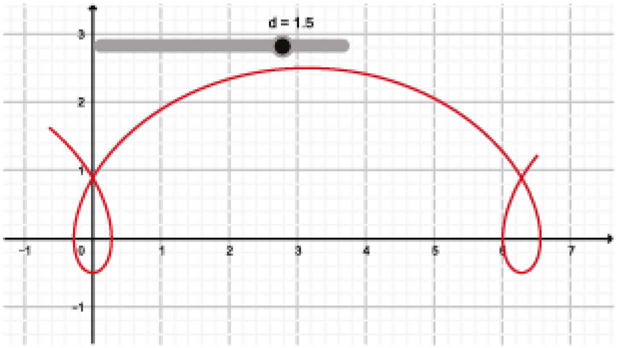


Fig. A11.5

9. To obtain the motion of a point along a prolate cycloid (d > 1) in the GeoGebra software, you need to:

1) create a slider a, varying from 0 to 2Pi;
2) construct a circle with center (a, 1) and radius d;
3) mark on it the point with coordinates (a-d*sin(a), 1-d*cos(a)) and connect it to the center of the circle with a segment;
4) in the "Input" bar, type: Curve(t-d*sin(t), 1-d*cos(t), t, 0, a) and press "Enter";
5) turn on the animation.

As a result, the circle will roll along the *x*-axis, and point C will trace a prolate cycloid (Fig. A11.6).

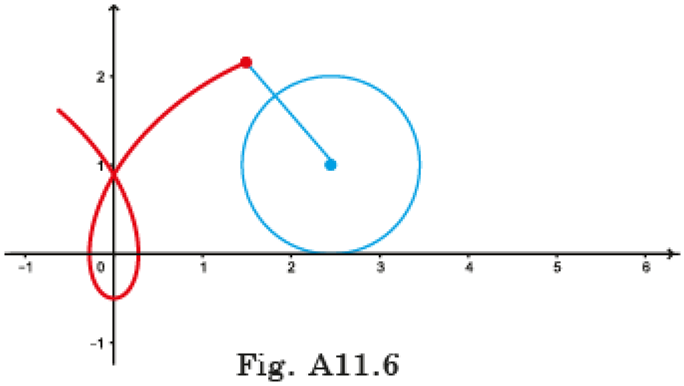


Fig. A11.6

10. Similarly to how it was done for the prolate cycloid, it can be shown that the parametric equations of the curtate cycloid (Fig. A11.7) are the equations

$$\begin{cases} x = t - d \cdot \sin t, \\ y = 1 - d \cdot \cos t. \end{cases}$$

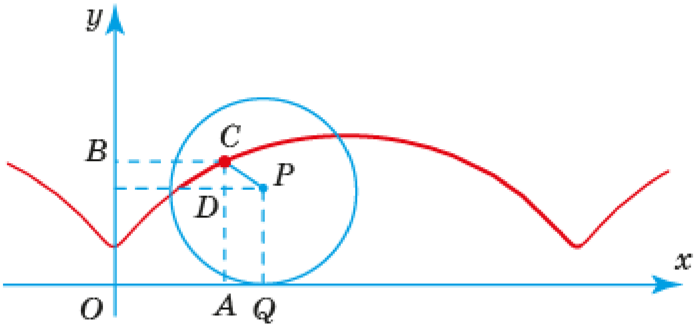


Fig. A11.7

To obtain a curtate cycloid in the GeoGebra software, you need to create a slider d, where 0 < d < 1. In the "Input" bar, type: Curve(t-d*sin(t), 1-d*cos(t), t, 0, 2Pi) and press "Enter". The result will be a curtate cycloid (Fig. A11.8).

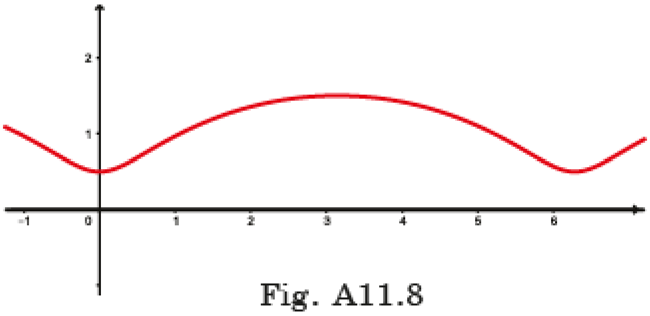


Fig. A11.8

11. To obtain the motion of a point along a curtate cycloid (0 < d < 1) in the GeoGebra software, you need to:

1) create a slider a, varying from 0 to 2Pi;

2) construct a circle with center (a, 1) and radius d;

3) mark on it the point with coordinates (a-d*sin(a), 1-d*cos(a)) and connect it to the center of the circle with a segment;

4) in the "Input" bar, type: Curve(t-d*sin(t), 1-d*cos(t), t, 0, a) and press "Enter";

5) turn on the animation.

As a result, the circle will roll along the x-axis, and the point will move along the curtate cycloid (Fig. A11.9).

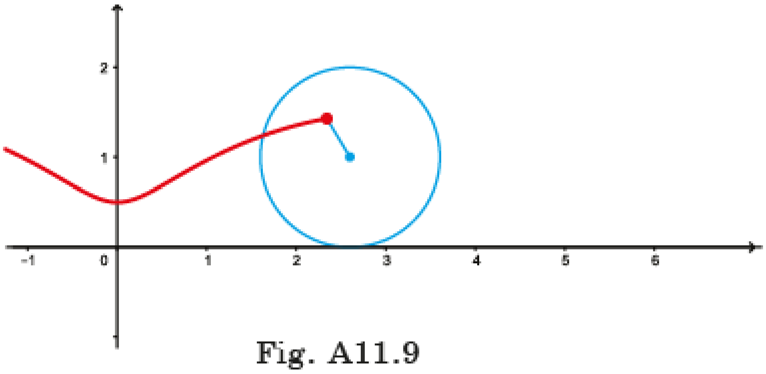


Fig. A11.9

12. To obtain a cardioid in the GeoGebra software, in the "Input" bar you need to type: Curve(2 cos t - cos 2t, 2 sin t - sin 2t, t, 0, 2Pi) and press "Enter". The result will be a cardioid (Fig. A11.10).

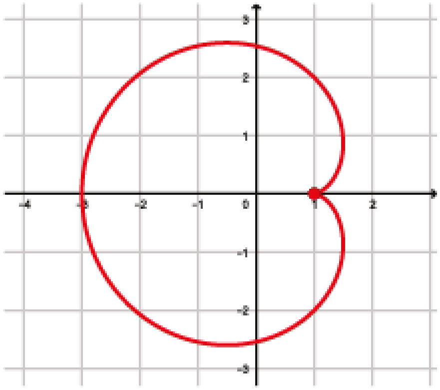


Fig. A11.10

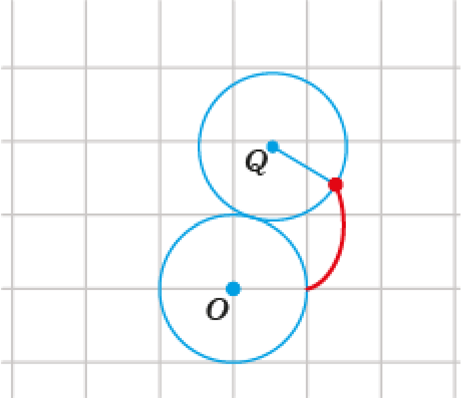


Fig. A11.11

13. To obtain the motion of a point along a cardioid you need to:

1) create a slider a, varying from 0 to 2Pi;

2) construct a circle with center (a, 1) and radius 1;

3) mark on it the point with coordinates ((2 cos a - cos 2a, 2 sin a - sin 2a)) and connect it to the center of the circle with a segment;

4) in the "Input" bar, type: Curve(2 cos t - cos 2t, 2 sin t - sin 2t, t, 0, a) and press "Enter";

5) Turn on the animation.

As a result, the circle with center Q will roll along the circle with center O, and point C will move along the cardioid (Fig. A11.11).

14. a) Figure A11.12. **15**. Figure A11.13.

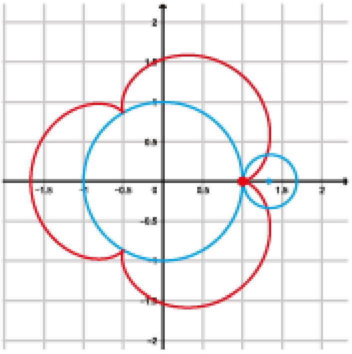


Fig. A11.12

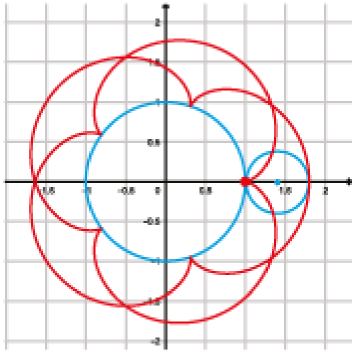


Fig. A11.13

16. Figure A11.14. **17**. Figure A11.15.

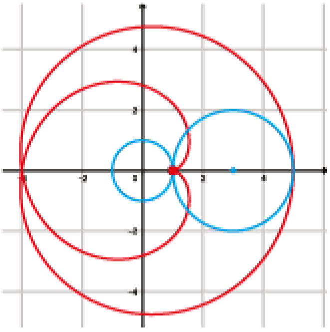


Fig. A11.14

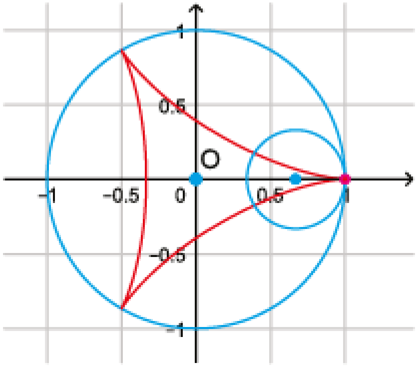


Fig. A11.15

18. Figure A11.16. **19**. Figure A11.17.

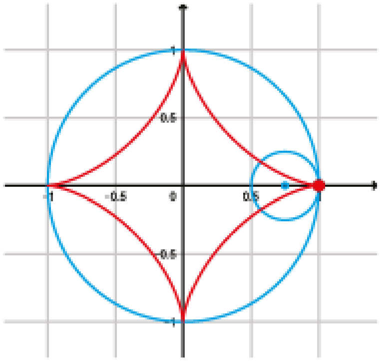


Fig. A11.16

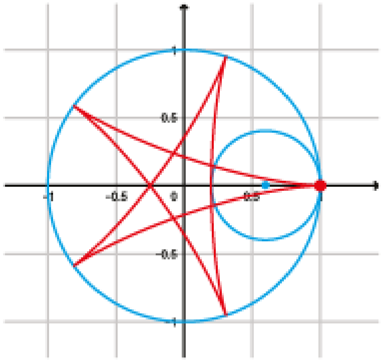


Fig. A11.17

20. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$. **21**. $\begin{cases} x = 2\cos t - d \cdot \cos(2t), \\ y = 2\sin t - d \cdot \sin(2t). \end{cases}$ If $d > 1$, the result is a prolate cardioid (Fig. A11.18). **22**. If 0 < d < 1, the result is a curtate cardioid (Fig. A11.19).

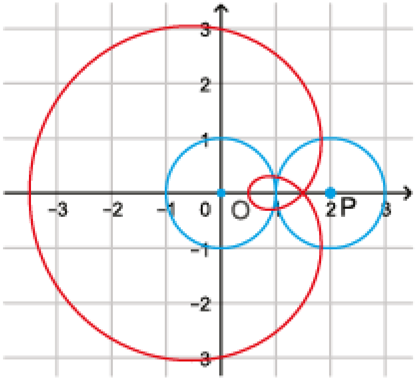


Fig. A11.18

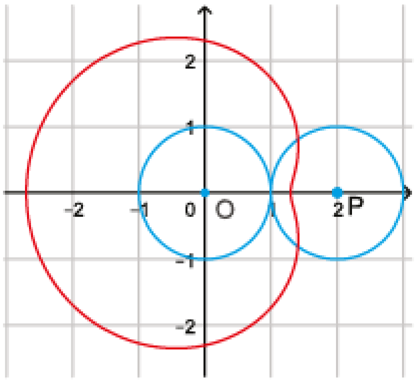


Fig. A11.19

23. $\begin{cases} x = \operatorname{ctg} t \pm c \cdot \cos t\,, \\ y = 1 \pm c \cdot \sin t\,. \end{cases}$ **24.** $\begin{cases} x = \cos t\,(2\cos t \pm c), \\ y = \sin t\,(2\cos t \pm c). \end{cases}$ **25.** $\begin{cases} x = 1 \pm \sin t\,, \\ y = \operatorname{tg} t\,(1 \pm \sin t). \end{cases}$

12

1. a) $(\frac{1}{2}, \frac{\sqrt{3}}{2})$; b) $(\sqrt{2}, -\sqrt{2})$. **2.** a) $(2, \frac{\pi}{4})$; b) $(10, \pi)$; c) $(2, -\frac{\pi}{3})$; г) $(2, \frac{5\pi}{6})$. **3.** Figer A12.11.

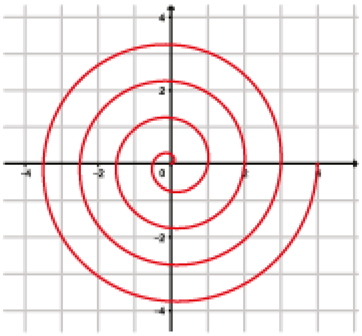


Fig. A12.1

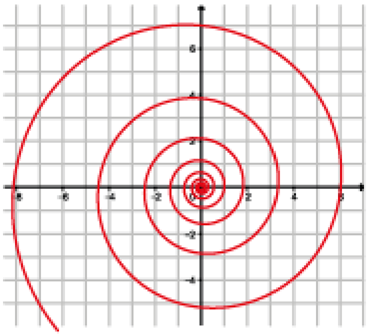


Fig. A12.2

4. Figer A12.1. **5.** $\begin{cases} x = t \cdot \cos t\,, \\ y = t \cdot \sin t\,. \end{cases}$ **6.** Figer A12.2. **7.** Figer A12.2. **8.** $\begin{cases} x = a^t \cdot \cos t\,, \\ y = a^t \cdot \sin t\,. \end{cases}$ **9.** Figer A12.3. **10.** Figer A12.4.

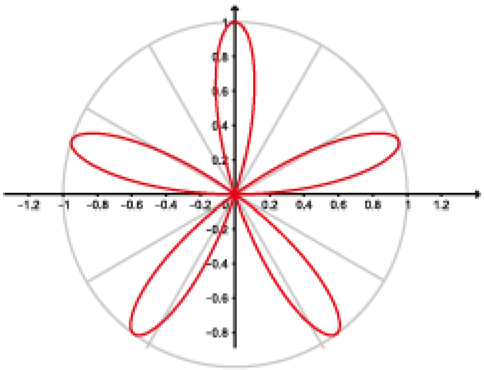


Fig. A12.3

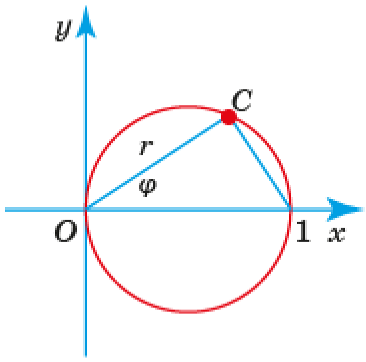


Fig. A12.4

11. Figer A12.5. **12.** Figer A12.6.

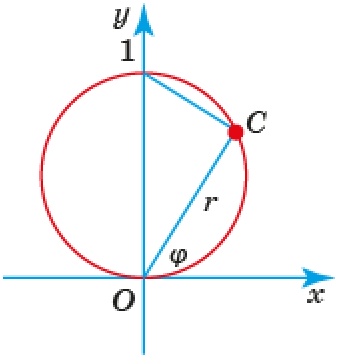


Fig. A12.5

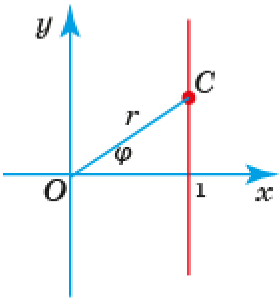


Fig. A12.6

13. Figer A12.7. **14.** Figer A12.8.

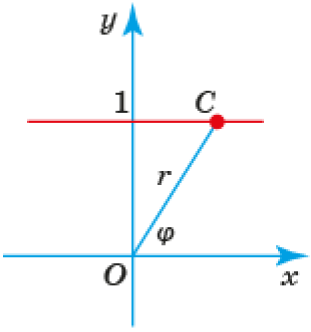


Fig. A12.7

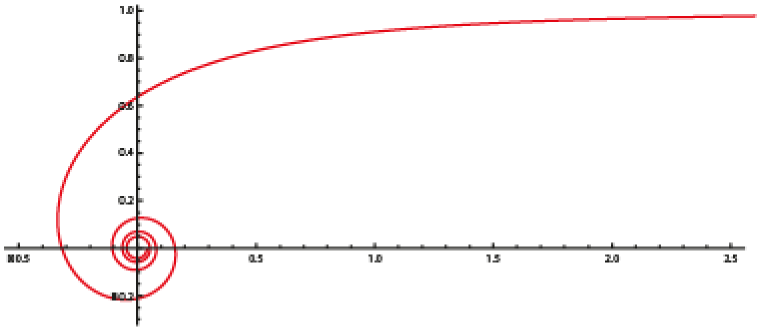


Fig. A12.8

15. Figer A12.9. **16.** Figer A12.10.

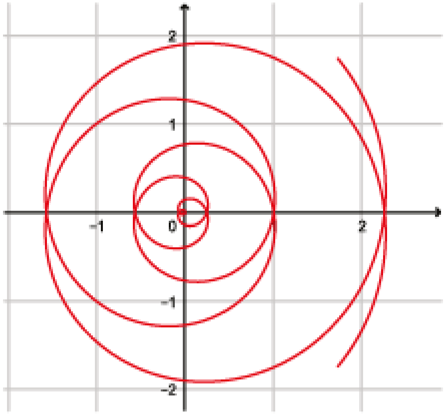


Fig. A12.9

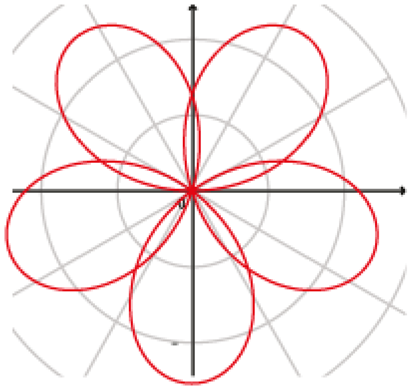


Fig. A12.10

17. Figer A12.11. **18.** Figer A12.12.

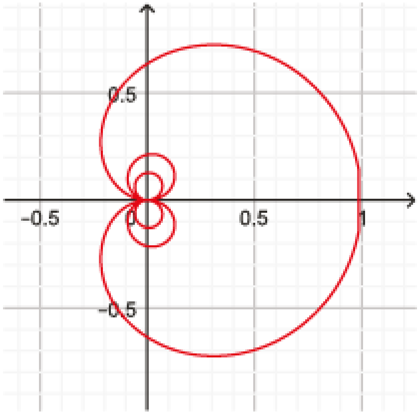


Fig. A12.11

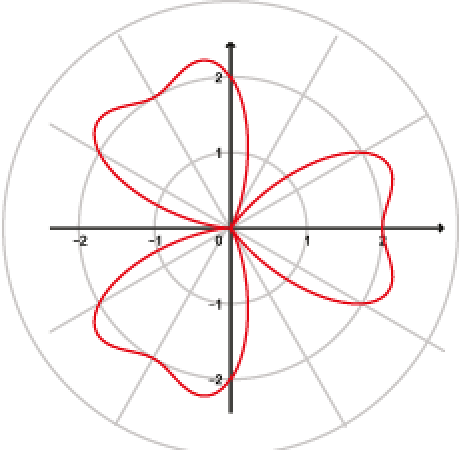


Fig. A12.12

19. Figer A12.13. **20.** The equality *CF* = *CD* (Fig. A12.14) can be rewritten as $r = 2a + r \cdot \cos\varphi$, from which the desired equation $r = \frac{2a}{1-\cos\varphi}$ follows.

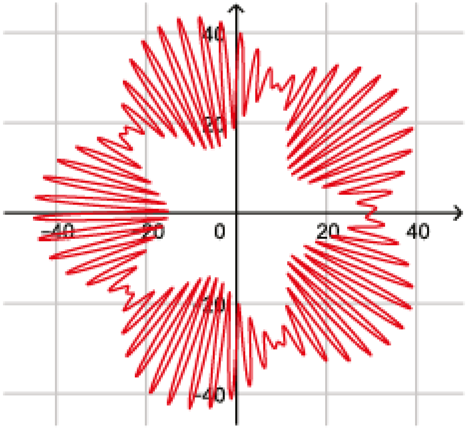


Fig. A12.13

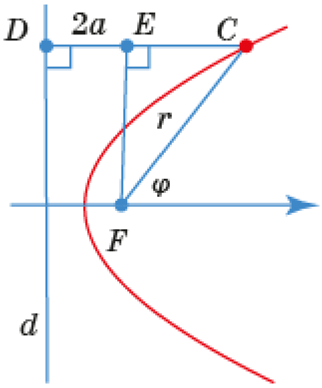


Fig. A12.14

21. Let us apply the law of cosines to triangle F_1F_2C (Fig. A12.15). We obtain the equality $(r-2a)^2 = 4c^2 + r^2 - 4cr \cdot \cos\varphi$, from which the desired equation $r = \frac{a^2-c^2}{a-c\cdot\cos\varphi}$ follows.

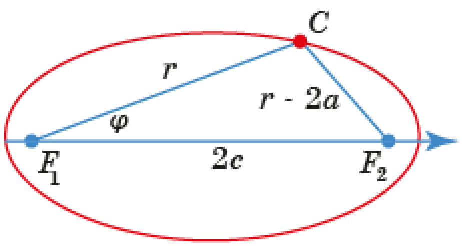


Fig. A12.15

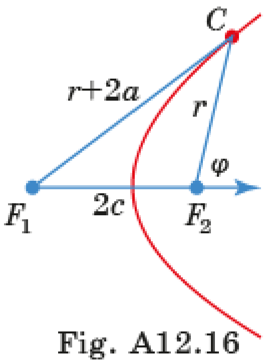


Fig. A12.16

22. Let us apply the law of cosines to triangle F_1F_2C (рис. A12.16). We obtain the equality $(r+2a)^2 = 4c^2 + r^2 + 4cr\cdot\cos\varphi$, from which the desired equation $r = \frac{c^2-a^2}{a-c\cdot\cos\varphi}$. **23.** a) parabola; b) ellipse; c) hyperbola. **24.** $r = d\frac{1\pm\sin\varphi}{\cos\varphi}$ (Fig. A12.17). **25.** $r = \frac{d\sin^2\varphi}{\cos\varphi}$ (Fig. A12.18).

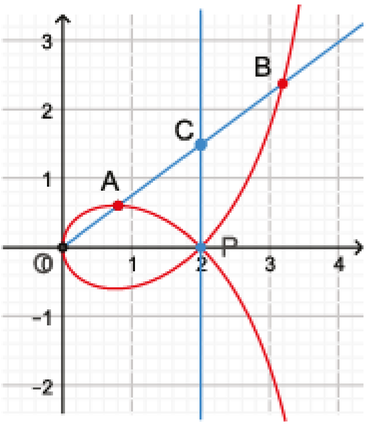


Fig. A12.17

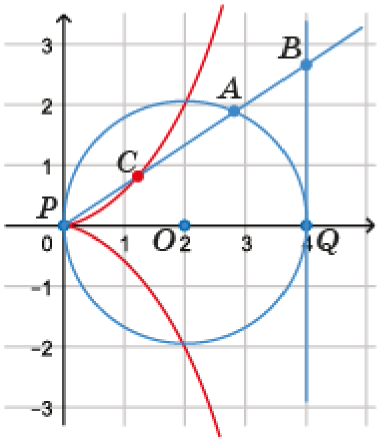


Fig. A12.18

26. Let us use the equation of Bernoulli's lemniscate in Cartesian coordinates $(x^2+y^2)^2 = 2a^2(x^2-y^2)$. Substitute $x = r\cdot\cos\varphi$, $y = r\cdot\sin\varphi$ into it. We obtain the desired equation $r^2 = 2a^2\cos 2\varphi$. **27.** By the law of sines applied to triangle ABC, we have the equality $\frac{r}{\sin 3\varphi} = \frac{d}{\sin 2\varphi}$, from which we obtain the desired equation $r = d\frac{\sin 3\varphi}{\sin 2\varphi}$ (Fig. A12.19).

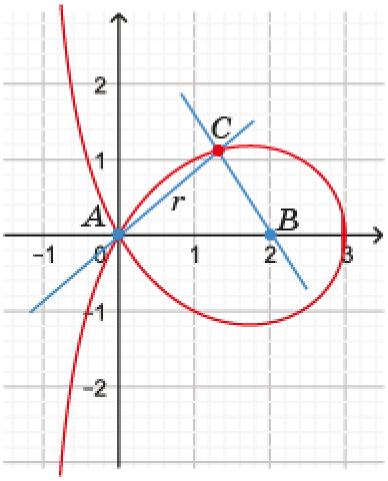


Fig. A12.19

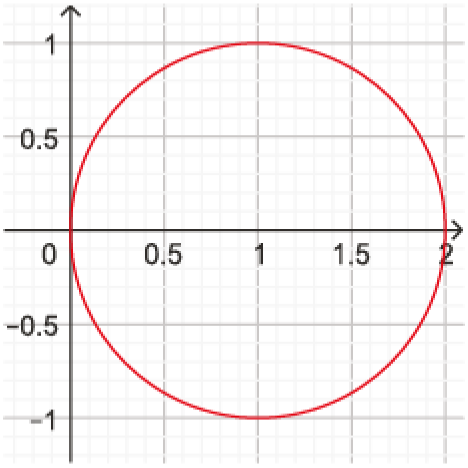


Fig. A12.20

28. $r = 2d \cdot \cos \varphi$, circle (Fig. A12.20). **29.** Let us use the equation the strophoid $r = d\frac{1\pm\sin\varphi}{\cos\varphi}$ (Fig. A12.17). Then $d\frac{1+\sin\varphi}{\cos\varphi} \cdot d\frac{1-\sin\varphi}{\cos\varphi} = d^2$. Hence under inversion with respect to a circle with center *O* and radius *d*, the strophoid maps onto itself. **30.** This parabola is given by the equation in polar coordinates $r = \frac{a}{\sin^2\frac{\varphi}{2}}$. Its inversion with respect to a circle with center *F* and radius *R* is given by the equation $r = \frac{R^2}{a} \cdot \sin^2\frac{\varphi}{2}$, which is the equation in polar coordinates of Pascal's snail (Fig. A12.21).

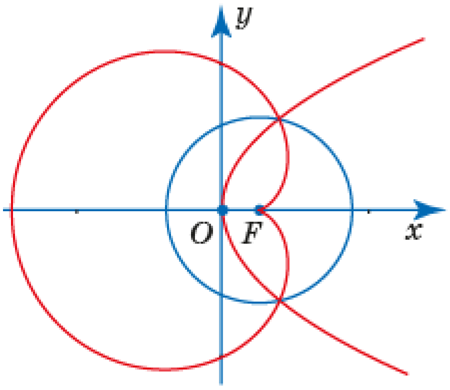


Fig. A12.21

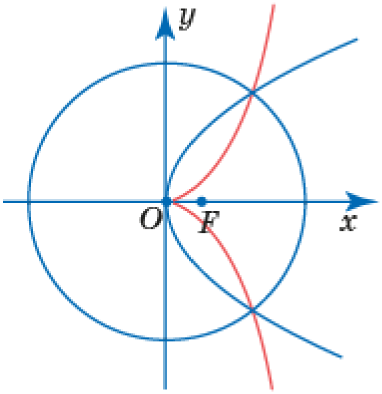


Fig. A12.22

31. This parabola is given by the equation in polar coordinates $r = \frac{4\,a\cdot\cos\varphi}{\sin^2\varphi}$. Its inversion with respect to a circle with center *O* and radius *R*, is given by the equation $r = \frac{R^2\sin^2\varphi}{4\,a\cdot\cos\varphi}$, which is the equation in polar coordinates of Diocles's cissoid (Fig. A12.22). **31.** This hyperbola is given by the equation in polar coordinates $r^2\cos 2\varphi = 1$. Its inversion is given by the equation $r^2 = \cos 2\varphi$, which is the equation in polar coordinates of Bernoulli's lemniscate. (Fig. A12.23).

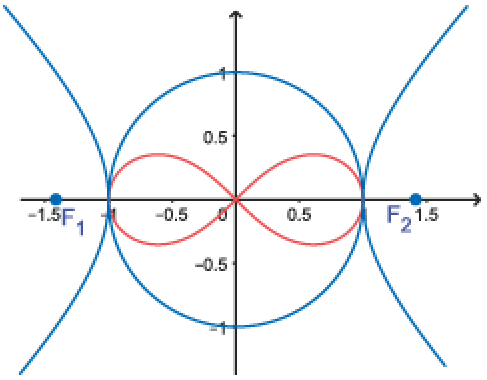


Fig. A12.23

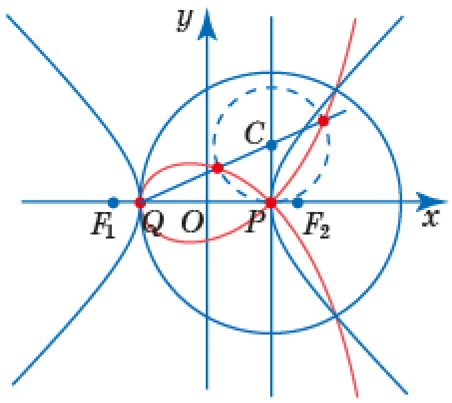


Fig. A12.24

32. Strophoid (Fig. A12.24). **33.** Pascal's snail (Fig. A12.25).

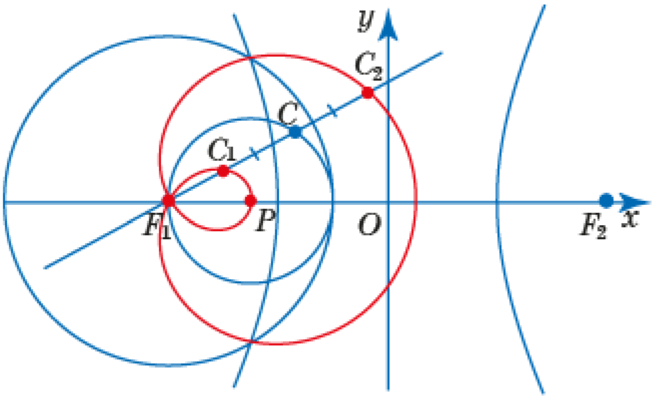


Fig. A12.25

13

2. Figer A13.1. **5.** Figer A13.2.

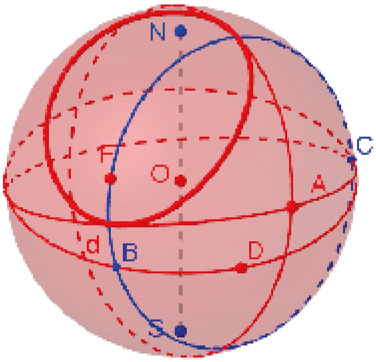


Fig. A13.1

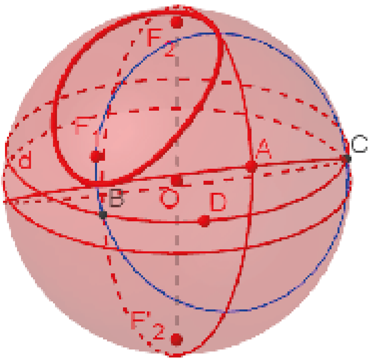


Fig. A13.2

8. Figer A13.3.

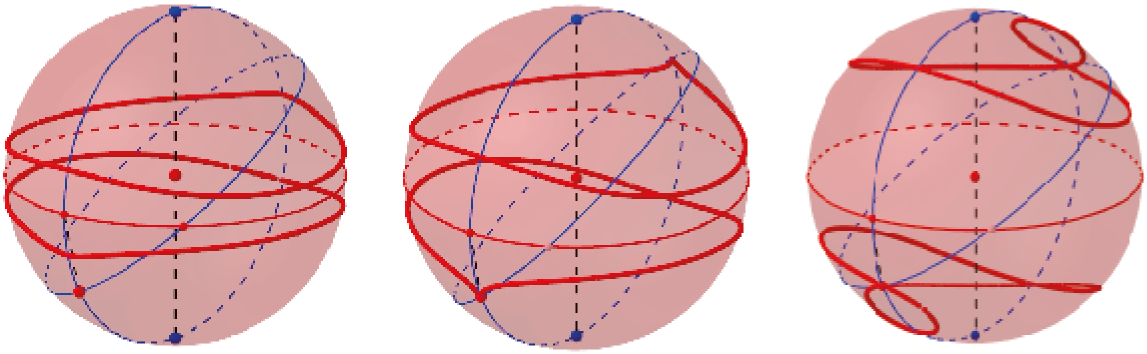
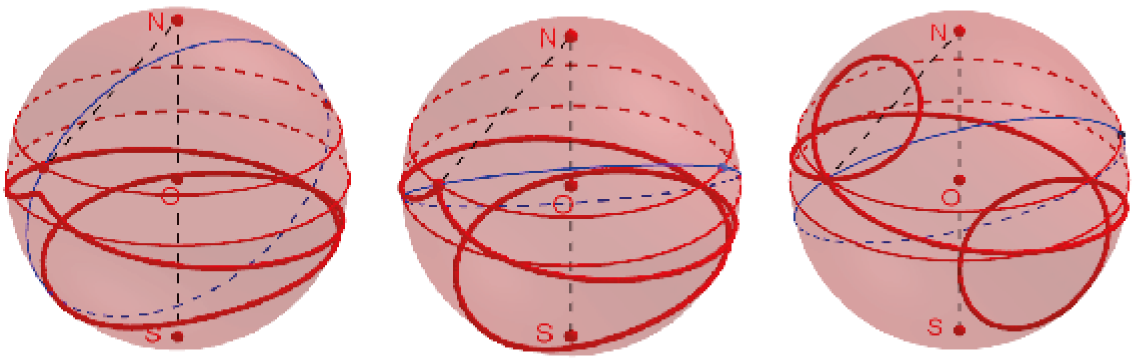

a) b) c)

Fig. A13.3

9. Figer A13.4.



a) b) c)

Fig. A13.4

14

11. Figer A14.1. **12.** Figer A14.2.

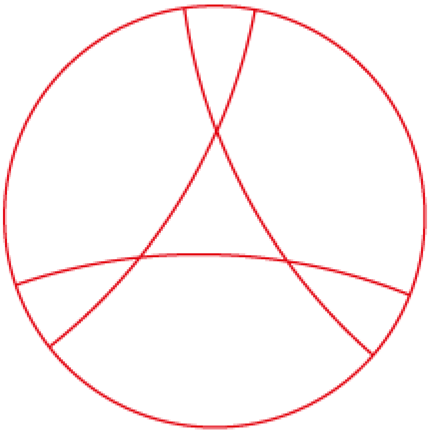


Fig. A14.1

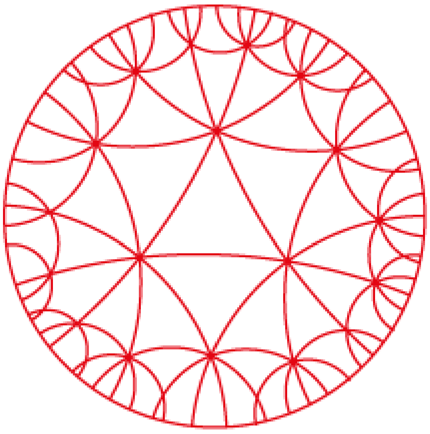


Fig. A14.2

13. Figer A14.3. **14.** Figer A14.4.

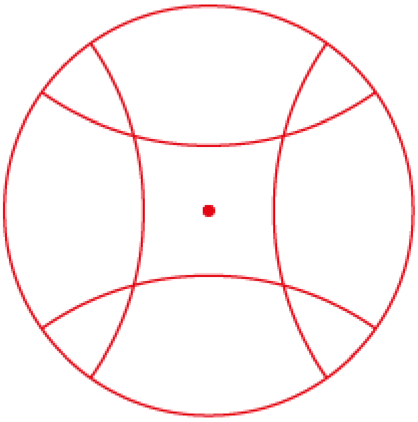


Fig. A14.3

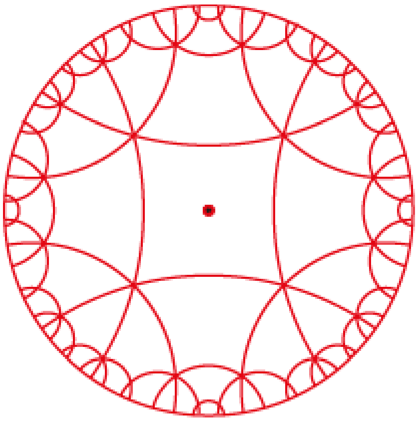


Fig. A14.4

15. Figer A14.5. **16.** Figer A14.6.

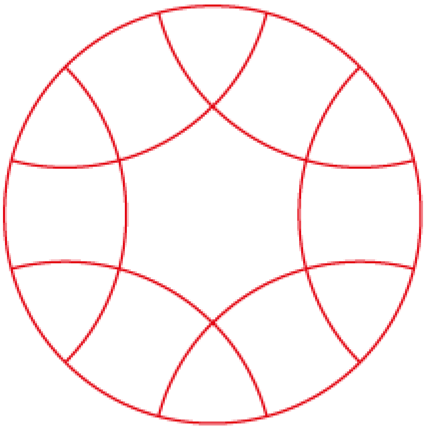


Fig. A14.5

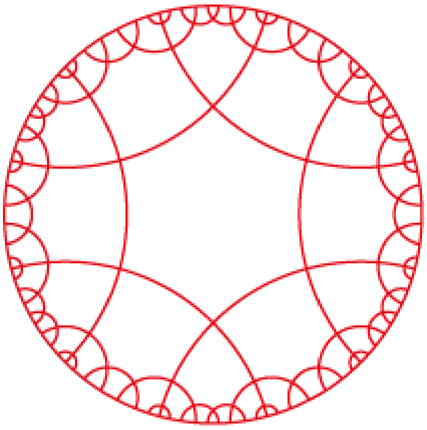


Fig. A14.6

CONTENT